In starting with the basic equation of transfer:

$$\mu \frac{d}{d\tau_\nu} I_\nu = I_\nu - S_\nu \rightarrow \mu \frac{dI}{d\tau} = I - S \quad \text{(e.g. integrating over } \nu) \quad (1)$$

we can <u>define</u>

$$M_n(\tau) = \int_{-1}^{1} I(\tau,\mu)\mu^n d\mu \quad (2)$$

for any suitable weighting (for example, if instead of $d\mu$ we use $dw(\mu)$ for some distribution over sampled angles). In this case we'll use uniform weighting. This means, for the plane parallel case ($\mu \neq \mu(\tau)$) that we can take successive <u>moments</u> of eq. (1),

$$\begin{aligned} \frac{d}{d\tau} M_1 &= M_0 - \int S\, d\mu && n=0 \\ \frac{d}{d\tau} M_2 &= M_1 - \int S\mu\, d\mu && n=1 \\ \frac{d}{d\tau} M_3 &= M_2 - \int S\mu^n d\mu && \text{ecc.} \end{aligned} \quad (3)$$

Here we make a choice regarding S: for the moment, take $S(\tau,\mu) \rightarrow S(\tau)$ only; with this statement, assuming isotropy, we have (trivially)

$$\int_{-1}^{1} \mu^n d\mu = \frac{2}{n+1}\, \delta_{n,(\text{even})} \quad \text{Kronecker } \delta$$

$$\frac{dM_1}{d\tau} = 2(M_0 - S), \quad \frac{dM_2}{d\tau} = M_1 \rightarrow \frac{d^2M_2}{d\tau^2} = 2(M_0 - S) \cdots, \quad (4)$$

and you see how the <u>closure problem</u> emerges. The system requires <u>some</u> relation among the moments. The way to do this is to introduce:

$$M_2 = f M_0, \quad \text{(\underline{Eddington factor})} \quad (5)$$

with $\underline{f}$ taken, perhaps, to be a function of τ. Then for f = constant,

$$\frac{f}{2}\frac{d^2M_0}{d\tau^2} \rightarrow \frac{d^2M_0}{dx^2} = M_0 - S, \quad dx = \left(\frac{2}{f}\right)^{1/2} d\tau$$

with the conditions that $M_0 \rightarrow S$ as $\tau \rightarrow \infty$ and $\tau \in [0,\infty)$.

⟹ The use of $\underline{f}$ in this approximation comes in from:

$$f \equiv \frac{\int I(\tau,\mu)\mu^2 d\mu}{\int I(\tau,\mu)\, d\mu}, \quad (6)$$

so you'll notice something <u>very</u> pretty: if you think of I as a distribution function for the radiation, and the medium is uniform - in the sense that the optical depth does not depend on μ - then, in effect:

$$f \sim \langle \mu^2 \rangle,$$

which is bounded $0 \leq f \leq 1$ so you can also think of $f^{1/2}$ as a sort of mean emergent direction cosine. So? This factor is a measure of the <u>anisotropy</u> of the radiation. If $f \rightarrow 1$, then we have freely streaming radiation. If $f \rightarrow \frac{1}{3}$, then we have the <u>Eddington approximation</u>

isotropic $\underset{M_2}{K} = \frac{1}{3}\underset{M_0}{J}$, so $3K - J$ is a useful way to

(<u>NB:</u> ~~for~~ f. is bound, in reality, by $\frac{1}{3} \leq f \leq 1$)

specify what happens if the light can emerge directly from the medium and is beamed.

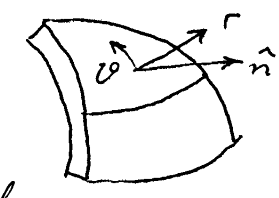
Now let's return to yesterday's lecture. For the most general form of the radiative transfer equation (RTE):

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \hat{n}\cdot\nabla\right) I_\nu(\underline{r}, t) = -\kappa_\nu \rho I_\nu + j_\nu \rho \qquad (7)$$

we will for now drop the time (steady state) [$\underline{c}$ is the speed of light] and write this in azimuthally symmetric form:

$$\cos\vartheta\frac{\partial}{\partial r} - \frac{\sin\vartheta}{r}\frac{\partial}{\partial\vartheta} \rightarrow \mu\frac{\partial}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial}{\partial\mu}$$

(a point from yesterday, I should have said that I'll use $\hat{n}$ where before I was using $\hat{k}$; my apologies). Now you see two things:



1. We can define the optical depth but only along radial coordinate paths, for unresolved objects we still want to remove μ,
2. the second term of the transformed gradient is due to the geometry and is a loss term:

$$\mu\frac{\partial I_\nu}{\partial r} = -\begin{pmatrix}\text{curvature}\\ \text{term}\end{pmatrix} - \begin{pmatrix}\text{absorption}\\ \text{term}\end{pmatrix} + \begin{pmatrix}\text{emission}\\ \text{term}\end{pmatrix} \qquad (8)$$

that vanishes as $r \to \infty$ (i.e. as $\frac{\Delta r}{r} \to 0$) for plane parallel.

In a purely absorbing medium, this states that the edges (limb) of the body will be darker than for the plane parallel case.

$\Rightarrow$ Already we have an important interpretation of stellar atmospheres, for a very extended atmosphere, in the absence of scattering (as we'll see) the medium looks darker at its edges than for a physically compact atmosphere. The consequences we will soon see.

So formally, we can do the same thing as before:

$$\int \mu^n \{\hat{n}\cdot\nabla I\} = -\kappa\rho \int \mu^n I + j\rho \int \mu^n$$

iff j is not a function of μ. Now:

$$\int \frac{1-\mu^2}{r}\mu^n \frac{\partial I}{\partial\mu} d\mu = \int \left(\frac{\mu^n - \mu^{n+2}}{r}\right)\left(\frac{\partial}{\partial\mu} I\right) d\mu$$

$$= \frac{1}{r}\int \frac{\partial}{\partial\mu}\left[(\mu^n - \mu^{n+2}) I\right] - \frac{1}{r}\int n\mu^{n-1} I + \frac{1}{r}\int (n+2)\mu^{n+1} I$$

$$= 0 - \frac{n}{r} M_{n-1} + \frac{n+2}{r} M_{n+1} \qquad (9)$$

why?

by our earlier definition of the moments. Oh, you say, this looks so very terrible! Now we have a coupled system of M_{n-1}, M_n, M_{n+1} where before we had only M_n, M_{n+1}. Yes, you're right!! It is worse, and this is because of the loss terms. Then

$$\int \mu^n d\mu = \frac{1}{n+1} \mu^{n+1} \Big|_{-1}^{1}$$

as before, or using S we have

$$(\text{rhs source term}) \rightarrow \frac{1}{n+1} \mu^{n+1} \Big|_{-1}^{1} S$$

so again if n is odd this vanishes; it is a small comfort, since now the curvature still couples the moments. So lets keep going:

$$\int \mu^{n+1} \frac{\partial}{\partial r} I = \frac{\partial}{\partial r} M_{n+1}$$

so we now have:

$$\frac{\partial M_{n+1}}{\partial r} + \frac{n+2}{r} M_{n+1} = \frac{n}{r} M_{n-1} - \kappa\rho M_n + j\rho \frac{1}{n+1} \delta_{n,\text{even}} \qquad (10)$$

essentially as before. We <u>still</u> need a closure condition, otherwise the system continues, but you can also see this is now starting to "smell" like something you've seen before, a tri-diagonal system, which you know from analysis (linear algebra): there's something here that – at least formally – looks like a finite difference of a vector $\underset{\sim}{M}$ in a diffusion-type system. I'm noting this for now only in a formal way, we'll develop the details soon, but for now:

$$A \underset{\sim}{M} = \underset{\sim}{V}$$

can be thought of as the most abstract form of the RTE (the matrix A is tridiagonal and infinite). If we know $\underset{\sim}{V}$ (which contains S) we can invert this at all orders and find $\underset{\sim}{M}(r)$. But, alas, we can't and that's why we must bring in lots of additional calculational machinery to treat this problem.