2/11/03

Line Transitions

When computing the statistical equilibrium populations, we wrote $\varphi(\nu)$ for the line profile function, which describes how – within the two level atom approximation – the radiation is distributed over the energy levels. (I'll switch throughout our discussion between λ and ν, so just keep in mind that $F_\lambda d\lambda = F_\nu d\nu$ and $d\lambda = -\frac{c}{\nu^2} d\nu$). So what is $\varphi(\nu)$?

First, knowing nothing about the intrinsic profile, we can immediately make some progress! Since λ_0 is a known quantity from the laboratory, call $\Phi_0(\lambda)$ the intrinsic profile function. Then, for a gas of atoms in thermal motion, we already know that along any line of sight $f(v)$ translates into:

$$\mathcal{G}(\nu) \sim e^{-(\Delta\nu/\Delta\nu_D)^2}$$

since the Doppler width is:

$$\Delta\nu_D = \frac{\nu_0}{c} v_T$$

and v_T is the velocity dispersion. In other words, any absorption or emission can occur at any frequency $\Delta\nu = \nu - \nu_0$ because of thermal motion. This means we have a convolution:

$$\varphi(\nu) = \mathcal{G} * \Phi_0 = \int_{-\infty}^{\infty} \mathcal{G}(\nu - \nu')\,\Phi_0(\nu')\,d\nu' = \int_{-\infty}^{\infty} \mathcal{G}(\nu')\,\Phi_0(\nu - \nu')\,d\nu'$$

since we've weighted the intrinsic profile by the probability of having an absorber at some velocity v relative to rest. If we have, for instance,

$$\Phi_0(\nu) = \Phi_0(0)\,\delta(\nu - \nu_0)\,, \quad \Phi_0(0) = \text{profile at line center}$$

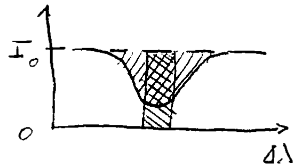
Then the profile function $\varphi(\nu)$ is a simple Gaussian in frequency with a width $\Delta\lambda_D$,

$$\varphi(\nu) = \Phi_0 \exp\left(-\left(\frac{\Delta\nu}{\Delta\nu_D}\right)^2\right) \rightarrow \tau_\nu = \tau_0\, e^{-\left(\frac{\Delta\nu}{\Delta\nu_D}\right)^2} \rightarrow \tau_\lambda = \tau_0\, e^{-\left(\frac{\Delta\lambda}{\Delta\lambda_D}\right)^2}$$

For the simplest case, the total opacity of the line (integrated over frequency) then grows linearly:

$$\int_{-\infty}^{\infty} (1 - e^{-\tau_\lambda})\,d(\Delta\lambda) \simeq \tau_0 \int_{-\infty}^{\infty} e^{-\left(\frac{\Delta\lambda}{\Delta\lambda_D}\right)^2} d(\Delta\lambda) \qquad (\tau_0 \ll 1)$$

Why did I write $1 - e^{-\tau_\lambda}$? This is the fraction of light that is absorbed, a quantity defined in the early days of spectroscopy called the equivalent width of the line (this is the width, in $\Delta\lambda$, that a perfectly dark profile would have regardless of the calibration of I_0, the background intensity – or flux)

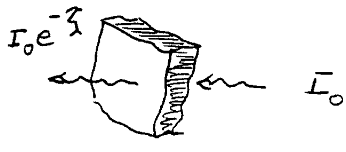


$$W_\lambda = \int_{-\infty}^{\infty} \left| \frac{I_\lambda - I_0}{I_0} \right| d(\Delta\lambda)$$

and this is a quantity defined for both emission and absorption. The case of a Gaussian profile is particularly simple since the integral is finite and we can write:

$$1 - e^{-\tau_\lambda} \simeq \tau_\lambda \quad \text{for } \tau_\lambda \ll 1 \text{ at all } \lambda.$$

By definition, this is a weak line. If we have a simple slab of gas, not thermally coupled to I_λ, through which the light passes, then the column density is proportional to τ_λ so $W_\lambda \sim \kappa_{\lambda_0} N_i$, where N_i is the column density of the lower level population. Since $\kappa_{\lambda_0} \sim B_{ij} g_i$, we have a way of getting the abundance of matter of different species directly. Well, not quite – the N_i would normally contain the excitation correction to go from $N_i \rightarrow N$, the actual abundance



2/11/03

of the ion (total abundance). But if $|i\rangle$ is $|0\rangle$, the *ground state*, then we don't have a temperature dependence and we can obtain N directly for any ion of known laboratory properties. The connection between B_{ij} and a laboratory measured value is to use the *oscillator strength*, f_{ij}, which is simply a convenient scaling of the measured value to the strength of a classical absorber, $\frac{\pi e^2}{mc} f_{ij} g_i$. From our definition of B,

$$B_{ij}\, g_i = B_{ij}\, g_j \rightarrow g_i f_{ij} = g_j f_{ji}$$

to connect absorbing and emitting states, and since you already know how to connect A and B, you see that measuring A_{ji} gives f_{ij} directly. We'll use this later.

NB We can add a simple assumption, that $\underset{\sim}{v}_T \rightarrow \underset{\sim}{v}_T + \underset{\sim}{v}_{turb} \equiv \underset{\sim}{v}_T + \underset{\sim}{\xi}$ (to use a conventional symbol) such that $\langle \underset{\sim}{\xi} \rangle = 0$, $\langle \xi^2 \rangle^{1/2} = \xi_t$, assumed usually to be a constant. This choice is the usual one for a Gaussian process and, since v_T and ξ_t are assumed to be independently distributed,

$$(\Delta\nu_D)^2 = \left(\frac{\nu_0}{c}\right)^2 (v_T^2 + \xi_t^2)$$

and for any temperature, the line width from the $\mathcal{G}(\nu)$ function is larger – that is, the line width is *superthermal* by some amount. This parameter, ξ_t, is called the *microturbulence*; to be blunt, without a physical guide to its origin(s), it is *just* a parameter of the radiative transfer, a guess! Take this statement seriously: look at how many physical *assumptions* you've made (gaussianity, independence, magnitude ...). I'll note now that the ξ_t appears to be required for nearly *all* astrophysical environments for which you have an independent determination of T, but calling this turbulence – instead of a stochastic velocity distribution – is a very big step and caution should be exercised. Perhaps the *only* place this is well founded is the interstellar medium, for which we have direct, and certain, observations of such motions. Even then, specifying the spectrum as Gaussian is problematic. The scale for ξ_t, the one that allows direct combination with v_T, must be of the same order as the mean free path for the gas; that's why the quantity is called *micro*turbulence. In reality, we should call $\mathcal{T}(\nu)$ the turbulent profile and take:

$$\mathcal{L}(\nu) = \mathcal{T}(\nu) * \mathcal{G}(\nu) * \Phi_0(\nu)$$

with $\mathcal{T}(\nu)$ determined from a separate hydrodynamical model (it *may* be Gaussian but not neccesarily, and at any rate won't enter $\Delta\nu_D$ in quadrature).

21/11/03

So to continue from today's session...

We now consider a two level system with a characteristic frequency ω_0, for resonance, and some factor we'll assume for damping. Then the equation of motion, now for a bound harmonic oscillator in some potential, is:

$$m_e \ddot{x} = -m_e \omega_0^2 x - \gamma \dot{x} + eE$$
$$\rightarrow \quad -\omega^2 x_\omega = -\omega_0^2 x_\omega - i\frac{\gamma}{m_e} x_\omega \omega + \frac{e}{m} E_\omega$$

since the equation is linear (no induced terms) and (ω_0, γ) are constants; hence, we can make a single mode expansion. Then

$$d_\omega \equiv e x_\omega = \frac{e}{m_e} E_\omega \Big/ \left[(\omega_0^2 - \omega^2) + i \frac{\gamma}{m_e} \omega\right]$$
$$= \frac{\frac{e}{m_e} E_\omega \left[(\omega_0^2 - \omega^2) - i \frac{\gamma}{m_e}\omega\right]}{\left[(\omega^2 - \omega_0^2)^2 + \frac{\gamma^2}{m_e^2}\omega^2\right]},$$

(NB: As I mentioned, although classical this must be the sam as the quantum result since the equation of motion results from Lagrangian that can be quantize

Now assume $\omega \approx \omega_0$. Then $\omega^2 - \omega_0^2 \sim 2\omega_0(\omega - \omega_0)$ so that:

$$d_\omega = \frac{\frac{e}{m_e} E_\omega \cdot 2\omega_0 \left((\omega - \omega_0) - i \frac{\gamma}{2m_e}\right)}{4\omega_0^2 \left[(\omega - \omega_0)^2 + \frac{\gamma^2}{4m_e^2}\right]}$$

and we take the imaginary part to be the opacity since you know from classical theory that the dipole moment is directly related to n, the index of refraction (notice that for $\omega = \omega_0$ the real part vanishes). Then calling $\gamma^2/4m_e^2 = \Gamma^2$, we can write

$$\Phi_0(\omega) = \frac{\Phi_0(0)}{(\Delta\omega)^2 + \Gamma^2} \qquad \text{(Lorentzian profile)}$$

as the intrinsic line profile. The width of a level is given by Γ, which must be related to A_{ji} as we discussed in class. Thus the profile we observe is the convolution of a Gaussian with this Lorentzian; written in dimensionless form normalized to $\Delta\omega_D$,

$$H(a,v) = \int_{-\infty}^{\infty} \frac{e^{-y^2} dy}{(v-y)^2 + a^2} = \int_{-\infty}^{\infty} \frac{e^{-(v-y)^2} dy}{y^2 + a^2} \qquad v = \frac{\Delta\omega}{\Delta\omega_D}$$

describes the total line profile ($a^2 = \frac{\Gamma^2}{\Delta\omega_D^2}$). The total, $H(a,v)$, is called the Voigt profile and this is the basic form for all our further treatment of line absorption.

NB Although for now I won't go into more detail on this function, you can easily see how it behaves by taking the integral as a convolution numerically. Numerous formal prescriptions are available for its integration; the essential point to recall is that as a increases, the wings (large $\Delta\omega \rightarrow$ large v) depart progressively from the Gaussian so even if the core saturates there will always be some absorption at large v. It just increases more slowly).

So at first, for small τ_0, the equivalent width of the profile (absorbed fraction of I_0) grows linearly, $W_\lambda \sim \tau_0$. For very large τ_0, if a is very small, the profile saturates (the line core is opaque and the wings don't grow rapidly) so W_λ increases only slowly with optical depth, roughly as $\ln \tau_0$. If, on the other hand, a is large (I mean $a \sim 0.1$ or so, so $\Delta E / k \Delta\omega_D \sim 0.1$, say) then because the profile varies as $(\Delta\omega)^{-2}$ we have $W_\lambda \sim \tau_0^{1/2}$. This phenomenon is referred to as a damping profile or as damped wings. You will encounter this especially in cosmological observations – the damped Ly α (DLA) systems seen by intergalactic absorption against very distant sources (e.g. QSO's – quasars).