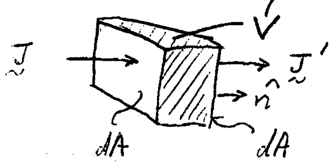
The Fluid Equations

Let's begin with a basic derivation. Imagine a fluid with a density ρ (mass density) moving with a velocity $\underset{\sim}{v}(\underset{\sim}{x},t)$. Then the <u>momentum flux</u> or <u>current</u> is

$$\underset{\sim}{J} = \rho \underset{\sim}{v}$$

and the flux passing through an area element dA along the normal is the mass rate of change:

$$\int \underset{\sim}{J}\cdot\hat{n}\, dA = \frac{dm}{dt}$$



so if dA is the same on both sides of the volume V and $\hat{n}$ is also the same, we mean the difference in the <u>mass flux</u> in compared to that exiting. Hence:

$$\int \underset{\sim}{J}\cdot d\underset{\sim}{A} = \int \nabla\cdot\underset{\sim}{J}\, dV \quad \text{but if } J_{in} > J_{out},\ \frac{dM}{dt} > 0$$

by Gauss' theorem and defining $M = \int \rho\, dV$ (of course) then:

$$-\frac{d}{dt}M = -\int \frac{\partial\rho}{\partial t} dV = \int \nabla\cdot\underset{\sim}{J}\, dV \therefore \frac{\partial\rho}{\partial t} + \nabla\cdot\underset{\sim}{J} = 0.$$

Obviously, if ρ = constant, J must be constant and $J_{out} - J_{in} = 0$ everywhere, or

$$\nabla\cdot\underset{\sim}{J} = 0 \rightarrow \rho\underset{\sim}{v}\cdot\hat{n} A = \text{constant},$$

the statement of mass conservation for a simple fluid. Thus if A changes, $\underset{\sim}{v}\cdot\hat{n}A$ will also change to maintain ρ = constant. This simple law is one you already know well from electrodynamics - where it's called charge conservation - and you also know that if $\nabla\cdot\underset{\sim}{J} = 0$ then $\underset{\sim}{J} = \nabla\times\underset{\sim}{\Psi}$, for some vector potential Ψ. Thus, as you also know from $\nabla\cdot\underset{\sim}{B} = 0 \rightarrow \underset{\sim}{B} = \nabla\times\underset{\sim}{A}$, you can write a general equation for the evolution of the flow in terms of a potential function. But for a fluid - or current - the motion may be continuous but the constituents are not, they're particles. So...

Assume we have a velocity distribution, $f(\underset{\sim}{v})$, for the particles. Then for any quantity,

$$Q(\underset{\sim}{x},t) = \int_{-\infty}^{\infty} Q(\underset{\sim}{x},\underset{\sim}{v},t)\, f(\underset{\sim}{v})\, d\underset{\sim}{v} \Big/ \int_{-\infty}^{\infty} f(\underset{\sim}{v})\, d\underset{\sim}{v}$$

with a range of random velocities $[-\infty,\infty]$; the new Q is now independent of $\underset{\sim}{v}$; we can assume $f(\underset{\sim}{v})$ is normalizable (so $f\rightarrow 0$ as $|\underset{\sim}{v}|\rightarrow\infty$). We can define

$$n(\underset{\sim}{x},t) \equiv \int_{-\infty}^{\infty} f(\underset{\sim}{v})\, d\underset{\sim}{v}$$

to be the normalizing factor ($N(t) = \int n(\underset{\sim}{x},t)\, d\underset{\sim}{x}$ is the total number of particles), the number density. Then:

$$n(\underset{\sim}{x},t)\, Q(\underset{\sim}{x},t) = Q(\underset{\sim}{x},t)\int f(\underset{\sim}{v})\, d\underset{\sim}{v}.$$

What we'll use, however, is a slightly different notation, $\langle Q\rangle$ for $Q(\underset{\sim}{x},t)$ after taking the velocity average. Thus:

$$n\langle\underset{\sim}{v}\rangle = \int f\underset{\sim}{v}\, d\underset{\sim}{v}$$

is the <u>mean velocity</u> and for particles of uniform mass m this becomes:

$$\underset{\sim}{J} = \rho\langle\underset{\sim}{v}\rangle = m\int f\underset{\sim}{v}\, d\underset{\sim}{v}.$$

Now we'll start with the evolution equation, <u>not</u> for the measurable quantities, but for f. It's reasonable to say that, in its most general form, $f\rightarrow f(\underset{\sim}{x},\underset{\sim}{v},t)$, subject to the condition that:

$$N(t) = \int f(\underset{\sim}{x},\underset{\sim}{v},t)\, d\underset{\sim}{x}\, d\underset{\sim}{v}.$$

Then, simply:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \underset{\sim}{v}\cdot\nabla f + \dot{\underset{\sim}{v}}\cdot\frac{\partial f}{\partial \underset{\sim}{v}} \quad \Rightarrow \text{Vlasov equation if } \frac{df}{dt}=0$$

$$= \frac{\partial f}{\partial t} + v_j\frac{\partial f}{\partial x_j} + \dot{v}_j\frac{\partial f}{\partial v_j}. \qquad \text{(summation over repeated index)}$$

If we assert that f doesn't change explicitly with any other variables, the basic assumption of equilibrium statistical mechanics, then

$$\frac{df}{dt} = 0, \qquad \text{(as I'd just written)}$$

so now taking $\int d\underset{\sim}{v}(\cdots)$ we have:

$$\int d\underset{\sim}{v}\,\frac{\partial f}{\partial t} + \int d\underset{\sim}{v}\,v_j\frac{\partial f}{\partial x_j} + \int d\underset{\sim}{v}\,\frac{\partial f}{\partial v_j}\dot{v}_j$$

$$= \frac{\partial}{\partial t}\int f\,d\underset{\sim}{v} + \int\left[\frac{\partial}{\partial x_j}(v_j f)\,d\underset{\sim}{v} - f\frac{\partial v_j}{\partial x_j}\right]d\underset{\sim}{v} + \int\left[\frac{\partial}{\partial v_j}(f\dot{v}_j) - f\frac{\partial \dot{v}_j}{\partial v_j}\right]d\underset{\sim}{v}$$

$$= \frac{\partial}{\partial t}n(\underset{\sim}{x},t) + \frac{\partial}{\partial x_j}(\langle v_j\rangle n) - n\left\langle\frac{\partial v_j}{\partial x_j}\right\rangle + f\dot{v}_j\Big|_{-\infty}^{\infty} - n\left\langle\frac{\partial \dot{v}_j}{\partial v_j}\right\rangle$$

$$= \frac{\partial}{\partial t}n(x,t) + \frac{\partial}{\partial x_j}n\langle v_j\rangle = 0$$

since:

$$\left\langle\frac{\partial v_j}{\partial x_j}\right\rangle = 0, \quad \left\langle\frac{\partial \dot{v}_j}{\partial v_j}\right\rangle = 0, \text{ and } f\to 0 \text{ at } \pm\infty \text{ by hypothesis.}$$

In other words, we have again the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\underset{\sim}{J} = 0, \quad (\text{multiplying by } \underline{m}),$$

$$\rightarrow \frac{\partial}{\partial t}M + \int\underset{\sim}{J}\cdot d\underset{\sim}{A} = 0.$$

This process now continues as we did for the RTE: we take the moments using as our independent variable $\underset{\sim}{v}$ and its powers ($v_i, v_iv_j, v_iv_jv_k$, etc.). OK, let's go. Take v_i:

$$v_i\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}(fv_i) - f\frac{\partial v_i}{\partial t} \rightarrow \frac{\partial}{\partial t}n\langle v_i\rangle - n\left\langle\frac{\partial v_i}{\partial t}\right\rangle$$

$$v_iv_j\frac{\partial f}{\partial x_j} = \frac{\partial}{\partial x_j}(fv_iv_j) - f\frac{\partial}{\partial x_j}(v_iv_j) \rightarrow \frac{\partial}{\partial x_j}n\langle v_iv_j\rangle - n\left\langle\frac{\partial}{\partial x_j}(v_iv_j)\right\rangle$$

and:

$$v_i\dot{v}_j\frac{\partial f}{\partial v_j} = \frac{\partial}{\partial v_j}fv_i\dot{v}_j - f\frac{\partial}{\partial v_j}(v_i\dot{v}_j).$$

First, since $\left\langle\frac{\partial v_i}{\partial t}\right\rangle = 0$ and $\left\langle\frac{\partial}{\partial x_j}(v_iv_j)\right\rangle = 0$ the first two equations give:

$$v_i\left(\frac{\partial f}{\partial t} + v_j\frac{\partial f}{\partial x_j}\right) \rightarrow \frac{\partial}{\partial t}(n\langle v_i\rangle) + \frac{\partial}{\partial x_j}(n\langle v_iv_j\rangle).$$

What does $\langle v_iv_j\rangle$ mean? If we separate the mean into ordered ($\langle v_i\rangle$) and random motion,

$$\langle v_iv_j\rangle = \langle(\langle v_i\rangle + u_i)(\langle v_j\rangle + u_j)\rangle$$

$$= \langle v_i\rangle\langle v_j\rangle + \langle v_i\rangle\langle u_j\rangle + \langle v_j\rangle\langle u_i\rangle + \langle u_iu_j\rangle$$

$$= \langle v_i\rangle\langle v_j\rangle + \langle u_iu_j\rangle$$

since $\langle u_i\rangle = 0$ by hypothesis in this separation. I'll explain more of this in class. Then:

$$\frac{\partial}{\partial x_j}n\langle v_iv_j\rangle = \frac{\partial}{\partial x_j}n(\langle u_iu_j\rangle + \langle v_i\rangle\langle v_j\rangle)$$

In general, we <u>define</u> $T_{ij} = n\langle v_iv_j\rangle$ to be the <u>stress tensor</u>, a quantity that describes the dynamics of the medium (it has the dimensions of a momentum flux) and we've separated $T_{ij} = T_{ij}^{ordered} + T_{ij}^{random}$, Now we make a very important assumption, that $\langle u_i u_j \rangle = \sigma^2 \delta_{ij}$ where as usual δ_{ij} is the unit tensor (matrix). Then:

$$T_{ij} = \rho \langle v_i \rangle \langle v_j \rangle + P\delta_{ij}$$

where P is the pressure. Consequently, the first terms give:

$$\frac{\partial}{\partial t} n\langle v_i \rangle + \frac{\partial}{\partial x_j}\left(n\langle v_i\rangle\langle v_j\rangle + \frac{P}{m}\delta_{ij}\right) \qquad \left(\text{NB} \quad \frac{\partial P}{\partial x_j}\delta_{ij} = \frac{\partial P}{\partial x_i}\right)$$

$$= \frac{\partial}{\partial t} n\langle v_i\rangle + \frac{\partial}{\partial x_j} n\langle v_i\rangle\langle v_j\rangle + \frac{1}{m}\frac{\partial P}{\partial x_i}.$$

Now what about $\int f v_i \dot{v}_j$ and related terms? why does this vanish?

$$\int v_i \dot{v}_j \frac{\partial f}{\partial v_j} d\underline{v} = \int d\underline{v} \left[\frac{\partial}{\partial v_j}(f v_i \dot{v}_j) - f\frac{\partial}{\partial v_j}(v_i \dot{v}_j)\right]$$

but:

$$\frac{\partial v_i}{\partial v_j} = \delta_{ij} \rightarrow \quad \dot{v}_j \frac{\partial v_i}{\partial v_j} + v_i \frac{\partial \dot{v}_j}{\partial v_j} = \dot{v}_i + v_i \frac{\partial \dot{v}_j}{\partial v_j}$$

The first term is simply the acceleration, but the second vanishes if $\dot{v}_j \neq \dot{v}_j(v_j)$, an assumed force law (clearly this fails if we have, say, a magnetic field). Thus if we define $\langle \dot{v}_i \rangle = n a_i$ then:

$$\frac{\partial}{\partial t} n\langle v_i\rangle + \frac{\partial}{\partial x_j}(n\langle v_i\rangle\langle v_j\rangle) + \frac{1}{m}\frac{\partial P}{\partial x_i} - n a_i = 0$$

Using the continuity equation gives:

$$n\left[\frac{\partial}{\partial t}\langle v_i\rangle + \langle v_j\rangle\frac{\partial}{\partial x_j}\langle v_i\rangle\right] = -\frac{1}{m}\frac{\partial P}{\partial x_i} + n a_i,$$

the equation of motion for the fluid. For a static fluid:

$$\frac{1}{\rho}\frac{\partial P}{\partial x_i} = a_i \qquad \text{hydrostatic equilibrium}$$

where we can now take, for gravity, $a_i = -\frac{GM}{r^2}\hat{r}$ and:

$$\frac{1}{\rho}\frac{\partial P}{\partial r} = -\frac{GM}{r^2}$$

(OK, as an aside, to show how far we can get:

$$\frac{d}{dr}\left(r^2 \frac{1}{\rho}\frac{\partial P}{\partial r}\right) = -G\frac{dM}{dr} = -4\pi r^2 G\rho \;\therefore \qquad \left(\text{use } \frac{dM}{dr} = 4\pi r^2 \rho\right)$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{1}{\rho}\frac{\partial P}{\partial r}\right) = -4\pi G\rho$$

is the equation of equilibrium for a self-gravitating compressible sphere, otherwise called the Lane-Emden equation; if we know $P(\rho)$ we can solve for $\rho(r)$ everywhere within an arbitrary volume). I know this digression appears odd now, but you'll see later that it must break down, we can start with a static model as a limit of the fluid equations but if we include radiation — the driving effect of radiation pressure — there is some radius at which the body cannot remain static.

So now we have two basic equations: