$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x_j} J_j = 0$$

$$\frac{\partial}{\partial t} J_i + \frac{\partial}{\partial x_j} T_{ij} + \frac{\partial P}{\partial x_i} = \rho a_i .$$

For steady state we have in one dimensional motion we have:

$$J = \rho v = \text{constant}$$
$$\rho v^2 + P = \text{constant}$$

(since for steady state we'll assume $a = 0$ for now). The latter is an equation you know, the Bernoulli equation for a streamline flow. In general, these hold if:

$$\underset{\sim}{J} \cdot \hat{n} = \text{constant}$$
$$T_{ij}\, \hat{n}_j = \text{constant}$$

, true across a surface of area A.

Now we can proceed to the last equation, the next moment

$$v_i v_j \left[\frac{\partial f}{\partial t} + v_k \frac{\partial f}{\partial x_k} + \dot{v}_k \frac{\partial f}{\partial v_k} \right] = 0$$

① $v_i v_j \frac{\partial f}{\partial t} \to \frac{\partial}{\partial t} n\langle v_i v_j\rangle + 0$

② $v_i v_j v_k \frac{\partial}{\partial x_k} f \to \frac{\partial}{\partial x_k}\left(n\langle v_i v_j v_k\rangle\right) + 0$

③ $v_i v_j \dot{v}_k \frac{\partial f}{\partial v_k} \to -\left[n\langle v_i \dot{v}_k \delta_{jk}\rangle + \langle v_j \dot{v}_k \delta_{ik}\rangle n\right]$

$$\to -\left[n\langle v_i \dot{v}_j\rangle + n\langle v_j \dot{v}_i\rangle\right]$$

But we need to consider how to deal with $\langle v_i v_j v_k\rangle$. You notice we have the same issue here that we encountered for the radiative transfer equation – the moments don't automatically complete a closed set of equations. We require some truncation rule. For a gas, this comes naturally since for a Gaussian the third moment ~~[illegible]~~ vanishes. Take:

$$\langle v^2 v_k\rangle \to \langle v^2\rangle\langle v_k\rangle \qquad (i = j)$$

so now we have:

$$\frac{\partial}{\partial t}\varepsilon + \frac{\partial}{\partial x_k} F_{\varepsilon,k} = -\rho \underbrace{a_k\langle v_k\rangle}_{\text{work term}}$$

defining

$$\varepsilon = \tfrac{1}{2}\rho\langle v^2\rangle$$
$$F_{\varepsilon,k} = \tfrac{1}{2}\rho\langle v^2\rangle\langle v_k\rangle = J_k h$$

and recalling our definition of the pressure:

$$h = \tfrac{1}{2}\langle v^2\rangle + \frac{P}{\rho}$$

is the <u>enthalpy</u> and the pressure is $\rho\langle u^2\rangle$ as before. Again, this form of the equations is especially useful since we can now include the next flux:

$$J \cdot \hat{n}\, h = \text{constant}.$$

and we've completed the basic set of both evolution equations <u>and</u> the conserved fluxes.

Aside on self-gravitating structures

NB This is not in the main direction of the course, but it will connect what we do here with Fisica Stellare and Astrofisica I.

Consider a body with spherical symmetry in hydrostatic equilibrium. Then as we just had from the condition that $\dot{v}_i = 0$,

$$\frac{1}{\rho}\nabla P = \nabla\Phi \rightarrow \nabla\cdot\frac{1}{\rho}\nabla P = \nabla^2\Phi$$

Poisson eq.

which becomes:

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\cdot\frac{1}{\rho}\frac{\partial P}{\partial r} = -4\pi G\rho.$$

But let me recapitulate what I'd last written:

$$\frac{dM(r)}{dr} = 4\pi G\rho(r)$$

$$\frac{1}{\rho(r)}\frac{d}{dr}P(r) = -\frac{GM(r)}{r^2}$$

Yes, these are the same (in fact, from the condition that $\Phi = \Phi_{grav}$ only, we get the Poisson equation) but now you can also see that:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{1}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

is only a function of $\underline{r}$ and $M(r) = \int_0^r 4\pi G\rho(r')dr'$. In the absence of any radiative loss, this configuration is adiabatic, we can find a formal equation for $P(r)$ and $\rho(r)$ from $P = P(\rho)$ only. Another way to say this, for any fluid, is that:

$$\nabla\times\left(\frac{1}{\rho}\nabla P\right) = \nabla\times\nabla\Phi = 0 \;\therefore\; \text{if } P = P(\rho) \text{ then}$$

$$\left[\nabla\left(\frac{1}{\rho}\right)\right]\times\nabla P = 0, \text{ called barotropic,}$$

i.e. surfaces of constant ρ are the same as those of constant P which are the same as equipotential surfaces since for $P(\rho) = F$:

$$\frac{dF}{d\Phi} = \text{constant}.$$

So now we ask, for $\frac{dP}{dr} = \frac{dP}{d\rho}\frac{d\rho}{dr} = F'\frac{d\rho}{dr}$, how can we write this equation for structure in a simple form? Suppose we write:

$$P = K\rho^{\alpha}, \quad \alpha = \text{constant}.$$

Then:

$$\frac{1}{\rho}\frac{dP}{dr} = K\alpha\rho^{\alpha-1}\frac{d\rho}{dr} \rightarrow \frac{\alpha K}{\alpha-1}\frac{d}{dr}\rho^{\alpha-1},$$

and defining

$$y = \rho^{\alpha-1} \rightarrow \rho = y^{\frac{1}{\alpha-1}}$$

then:

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{dy}{dr} = -\frac{4\pi G}{K}\frac{\alpha-1}{\alpha}y^{\frac{1}{\alpha-1}} \qquad n \equiv \frac{1}{\alpha-1}$$

$$= -\frac{4\pi G}{K}\frac{1}{n+1}y^n. = -\lambda^{-2}y$$

One more step: define

$$\lambda^{-2} = \frac{4\pi G}{(1+n)K} \quad \text{and} \quad r = \lambda x, \text{ then}$$

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) = -y^n,$$

subject to the boundary condition that:

$$y(0) = 1 \,, \quad y'(0) = 0$$

(maximum density at the center). This is the Lane-Emden equation of index n and we see that we now have a way - in general - of computing $y(x) \rightarrow \rho(r)$ for any assumed equation of state of the form $P = K\rho^{\alpha}$.

NB : Why talk about this, especially now in these notes? This was the first stellar model, by Emden, Eddington, and Chandrasekhar - before 1938, with the advent of nuclear astrophysics - but there's another reason. For any gas in gravitational equilibrium, this model is true. For instance, if we have a cluster, a collection of stars in bound orbits, this can be used to model the mass. For dynamically bound masses, we don't need to consider radiation (well, except gravitational waves and those don't matter!).

For the mass, we have:

$$m(r) \sim \int_0^r r^2 \rho \, dr \sim \int_0^r \frac{d}{dr}\left(r^2 \frac{1}{\rho}\frac{dP}{dr}\right) dr$$

$$\sim - \int_0^x \frac{d}{dx}(x^2 y')\,dx \rightarrow x^2 y' \Big|_0^x \,,$$

but what is the total mass?

You see the unusual nature of the structure problem. Although you know the conditions at $x = 0$, you don't know where the surface occurs! We can only find this by solving for $y(x)$, but then

$$y_n(x_{*,n}) = 0$$

defines the surface - yet again, it occurs where ρ, and therefore P, vanishes. Thus

$$m(x_{*,n}) = -x^2 y_n' \big|_{x_{*,n}}$$

is, by construction, the scaled total mass of this configuration. In one sense, this exercise is circular. From the mass and hydrostatic equations, you already know the solution. But in another sense, it is a profound result: once you know the equation of state, $P(\rho)$, all masses that obey this have identical internal structure simply scaled by the central pressure and density. While the Lane-Emden equation is completely unrealistic for the modeling of stars, its original purpose, it provides an excellent starting model for a full solution of the structure equations. And it also works as a good approximation for degenerate configurations such as white dwarfs.