Let's look for a bit longer at the conservative forms of the fluid equations. They have the general form:

$$\frac{\partial}{\partial t}\begin{pmatrix}\text{some}\\ \text{density}\end{pmatrix} + \nabla\cdot\begin{pmatrix}\text{some}\\ \text{flux}\end{pmatrix} = \begin{pmatrix}\text{some force,}\\ \text{some source}\end{pmatrix}.$$

The fluxes are either vectors (mass, energy) or tensors (momentum). You see, then, why these can be ~~all~~ written in invariant form so easily; in relativity, calling x^0 the time dependent coordinate and choosing η_{ij} as the metric ($a\cdot b = a^0 b^0 - \underset{\sim}{a}\cdot\underset{\sim}{b}$ for all (a^i, b^i)) means we have $T^{00} = \rho$ and $T^{ij} = \rho\delta^{00} - (\rho\langle v^i\rangle\langle v^j\rangle + P\delta^{ij})$. In fact, P is then seen as an energy density in the non-ordered motions, so $\langle u_i u_j\rangle$ is (in Cartesian coordinates) is the general expression for what we took to be an isotropic term. This is also the case for a magnetic field, the reason for this digression.

The term that enters the pressure is a scalar quantity, one for which the force results from taking a gradient. In other words, you have two parts to T_{ij}, one that is not isotropic and one that is; the first produces a force by taking the divergence $\partial_j S_{ij}$, writing:

$$T_{ij} = S_{ij} + T\delta_{ij}$$

Now, for a magnetic field in the fluid limit, the force is $\frac{1}{c}\underset{\sim}{J}\times\underset{\sim}{B}$. Then using the Maxwell equations,

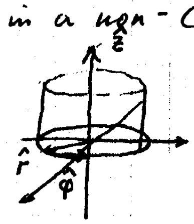
$$\frac{1}{4\pi}\nabla\times\underset{\sim}{B} = \underset{\sim}{J}$$
$$\nabla\cdot\underset{\sim}{B} = 0$$

so the equation for the force is $\underset{\sim}{J}\times\underset{\sim}{B} \rightarrow (\nabla\times\underset{\sim}{B})\times\underset{\sim}{B}$. We'll return to this in more detail soon but for now, notice that this produces a stress tensor

$$M_{ij} = \frac{1}{4\pi}(B_i B_j - \tfrac{1}{2}B^2\delta_{ij})$$

so we have one term that is of the S_{ij} form and another varying as $T\delta_{ij}$. You know that this second term is an energy density of the field, and P is for the kinetic energy in the random motions. Thus you see we have an energy density that takes the name <u>pressure</u> even though it's really the reaction of the medium on compression. For $B_i B_j$, you have something else - and this is the same for S_{ij} - a term that changes if you alter the geometric arrangement of the field. This holds for S_{ij} because if you twist the streamlines, $\langle v_i\rangle\langle v_j\rangle$, you're still doing work even if the total energy in the field remains the same in magnitude.

OK, so let's look at what we have from the equation of motion in a non-Cartesian coordinate system. For a cylindrical system the line element is:

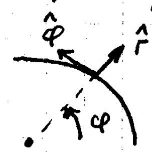


$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$
$$= dr^2 + r^2 d\varphi^2 + dz^2$$

so for the gradient:

$$\nabla \rightarrow \hat{r}\frac{\partial}{\partial r} + \hat{\varphi}\frac{1}{r}\frac{\partial}{\partial\varphi} + \hat{z}\frac{\partial}{\partial z}$$

Now we can proceed formally or simply to look at the term $\underset{\sim}{V}\cdot\nabla\underset{\sim}{V}$. I'll go simply, of course. Taking $\hat{r}$ and moving it along $\hat{r}$, nothing changes; the same is true for $\hat{\varphi}$. So:



$$\left(V_r\frac{\partial}{\partial r} + V_\varphi\frac{1}{r}\frac{\partial}{\partial\varphi}\right)(V_r\hat{r} + V_\varphi\hat{\varphi})$$
$$= \left(V_r\frac{\partial}{\partial r}(V_r\hat{r}) + V_r\frac{\partial}{\partial r}(V_\varphi\hat{\varphi})\right) + \left(\frac{1}{r}V_\varphi\frac{\partial}{\partial\varphi}(V_r\hat{r}) + \frac{1}{r}\frac{\partial}{\partial\varphi}(V_\varphi\hat{\varphi})\right)$$
$$= \left(V_r\frac{\partial V_r}{\partial r}\right)\hat{r} + \left(V_r\frac{\partial}{\partial r}V_\varphi\right)\hat{\varphi} \quad \downarrow$$
$$+ \left(\frac{V_r}{r}V_\varphi\frac{\partial\hat{r}}{\partial\varphi}\right) + \left(\frac{1}{r}V_\varphi\frac{\partial V_r}{\partial\varphi}\right)\hat{r} + \left(\frac{V_\varphi^2}{r}\frac{\partial\hat{\varphi}}{\partial\varphi}\right) + \left(\frac{1}{r}V_\varphi\frac{\partial V_\varphi}{\partial\varphi}\right)\hat{\varphi}$$

since the change in $\hat{r}$ is along the $\hat{\varphi}$ direction and that of $\hat{\varphi}$ is along the $-\hat{r}$ direction. Thus you have:

$$\rho\left(\frac{\partial}{\partial t}V_r + V_r\frac{\partial}{\partial r}V_r - \frac{V_\varphi^2}{r}\right)\hat{r} + \rho\left(\frac{\partial V_\varphi}{\partial t} + \frac{1}{r}V_rV_\varphi + V_r\frac{\partial}{\partial r}V_\varphi\right)\hat{\varphi}$$
$$+ \rho\left(V_\varphi\frac{1}{r}\frac{\partial V_\varphi}{\partial \varphi}\right)\hat{\varphi} + \rho\left(\frac{1}{r}V_\varphi\frac{\partial V_r}{\partial \varphi}\right)V_r$$

for the collected terms. Notice that: $\qquad V_j \equiv \langle v_j \rangle$

$$\frac{1}{r}V_rV_\varphi + V_r\frac{\partial V_\varphi}{\partial r} = V_r\cdot\frac{1}{r}\left[V_\varphi + r\frac{\partial V_\varphi}{\partial r}\right]$$
$$= \frac{1}{r}V_r\frac{\partial}{\partial r}(rV_\varphi)$$

so you see how angular momentum enters the equations of motion. For <u>axisymmetric</u> motion, $\partial/\partial\varphi \to 0$. Also, I've ignored $\hat{z}$ but you easily see how – because this direction is orthonormal to $(\hat{r}, \hat{\varphi})$, it remains separate (angular momentum changes for flows in the plane but not along the perpendicular direction) so:

$$\rho\left(\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + V_z\frac{\partial V_r}{\partial z}\right) + \frac{\partial P}{\partial r} = a_r\rho \quad \text{now includes } \frac{V_\varphi^2}{r}\rho$$
$$\rho\left(\frac{\partial V_\varphi}{\partial t} + \frac{1}{r}V_r\frac{\partial}{\partial r}(rV_\varphi) + V_z\frac{\partial V_\varphi}{\partial z}\right) + \underbrace{\cancel{\frac{1}{r}\frac{\partial P}{\partial \varphi}}}_{=0} = \underbrace{\cancel{a_\varphi\rho}}_{=0 \text{ by assertion}}$$
$$\rho\left(\frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r} + V_z\frac{\partial V_z}{\partial z}\right) + \frac{\partial P}{\partial z} = a_z\rho$$

For time independent flows, then, rV_φ = constant if $V_\varphi \neq V_\varphi(z)$ and for purely planar flows:

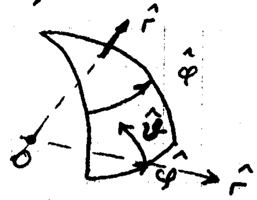
$$\frac{\partial P}{\partial z} = a_z$$

separately. This, in other terms, is how you build a disk – the $\hat{z}$ flows vanish in equilibrium because it's planar symmetric.

I won't now generalize to an arbitrary coordinate system but you see how this would work for spherical coordinates:

$$ds^2 = dr^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta\, d\varphi^2$$

because of the projection of circles at constant ϑ.



To return to the fluid equations, for a specific equation of state, i.e. a barotropic formulation, we have:

$$\frac{1}{\rho}\frac{\partial P}{\partial x_i} = \frac{1}{\rho}\frac{\partial P}{\partial \rho}\frac{\partial \rho}{\partial x_i}$$

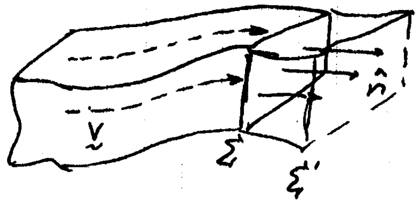
which you see gives – for a general form $P = K\rho^\gamma$,

$$-\frac{1}{\rho}\frac{\partial P}{\partial x_i} = \frac{\gamma}{\gamma-1}\frac{\partial}{\partial x_i}\left(\frac{P}{\rho}\right)$$

so for the enthalpy, noting that $c_s^2 = \left(\frac{\partial P}{\partial \rho}\right)_s$ (we'll prove this momentarily), we have:

$$h = \frac{1}{2}V^2 + \frac{\gamma}{\gamma-1}\frac{P}{\rho}.$$

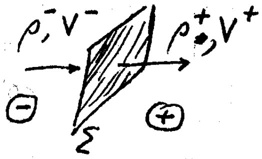
So the general conserved fluxes along the normal to a streamline are:



$$\left.\begin{aligned} \rho \underset{\sim}{V}\cdot\hat{n} &= \text{constant} = C_1 \\ \rho(\underset{\sim}{V}\cdot\hat{n})^2 + P &= \text{constant} = C_2 \\ \frac{1}{2}V^2 + \frac{c_s^2}{\gamma-1} &= \text{constant} = C_3 \end{aligned}\right\} \text{ for 1D motion } \underset{\sim}{V} = V\hat{n}$$

Now we imagine placing an arbitrary surface Σ across the flow. Obviously Σ has some area – call this A_Σ – and the normal to Σ is $\hat{n}_\Sigma$. We'll assume 1D flow through this area and that:

$$C_1^- = C_1^+ \rightarrow \rho^- V^- = \rho^+ V^+,$$



for example. Instead, let's write $(\rho_0, V_0, P_0, \ldots)$ for the $\ominus$ side and $(\rho_1, V_1, P_1, \ldots)$ for the $\oplus$ side. Since Σ is arbitrary, we have:

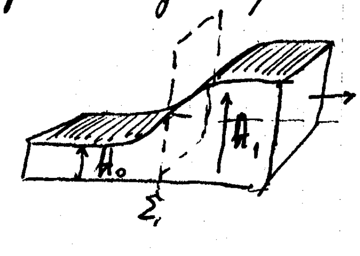
$$\rho_0 V_0 = \rho_1 V_1 \qquad \text{(for any fluid!)}$$
$$\rho_0 V_0^2 + P_0 = \rho_1 V_1^2 + P_1 \qquad \text{(for any eq. of state!!)}$$
$$\frac{1}{2}V_0^2 + \frac{\gamma}{\gamma-1}\frac{P_0}{\rho_0} = \frac{1}{2}V_1^2 + \frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1} \qquad (\text{for } P \sim \rho^\gamma)$$

Notice that we no specific Σ in mind, all variables can be continuous across Σ. But it may happen that some quantity changes, for instance the density (compressible) or the velocity or the pressure may change. So you know always:

$$V_1 = \frac{\rho_0}{\rho_1}V_0$$

which is why, for an incompressible medium, the velocity doesn't support density changes.

NB Simply, the change in the level of a fluid, H, is the analog conserved quantity to the density. For water, for example, the phenomenon of a tidal bore, or the formation of a "wall" around fluid coming from a tap on a sink (or the formation of walls of water in rapids and toilets),



For instance, Σ doesn't need to be infinitely thin – it could be a region, in which case the equations are equivalent to a set of finite difference equations. You can always write $\rho_1 \rightarrow \rho_0 + \delta\rho$, $P_1 \rightarrow P_0 + \delta P$, $V_1 \rightarrow V_0 + \delta V$ and then solve for the unknown quantities. This is in effect what you would do when

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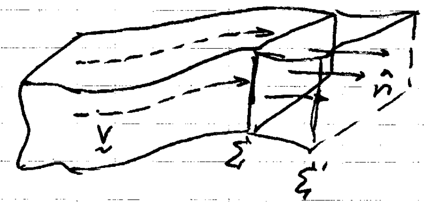

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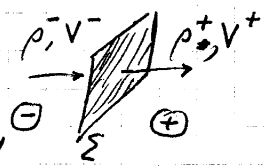

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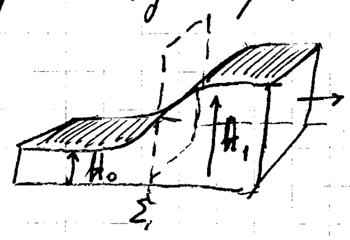

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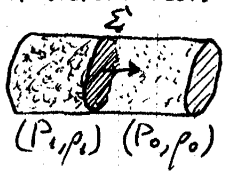
But we already have a fundamental result at hand: we can treat <u>any discontinuous</u> change along the flow!! Since (C_1, C_2, C_3) are constants, it follows that even for a region of infinitesimal thickness, one where <u>any</u> of the three fundamental variables jumps, we can find a solution for the other two — this is the solution for a <u>shock</u>.

Shocks in Compressible Fluids

Although this would normally come much later for the usual treatment of a fluid, we can look <u>here</u> at what happens. The tools are all assembled. Take the case where a diaphragm (i.e. a thin surface separating two gases) ruptures (i.e. it breaks at some predetermined $\Delta P = P_1 - P_0$). Now from the left, a compression "wave" moves to the right. In the frame moving with Σ, now a "surface", the gas has a velocity v_0. Then:



$$\rho_0 v_0 = \rho_1 v_1$$
$$\rho_0 v_0^2 + P_0 = \rho_1 v_1^2 + P_1$$
$$\frac{1}{2} v_0^2 + \Gamma \frac{P_0}{\rho_0} = \frac{1}{2} v_1^2 + \Gamma \frac{P_1}{\rho_1} \quad , \quad \Gamma = \frac{\gamma}{\gamma-1}$$

where for an ideal gas $\gamma = C_p/C_v$, the ratio of specific heats. Note that this will apply for any choice of equation of state of the form $P \sim \rho^\gamma$ so we'll use the terms barotropic or <u>polytropic</u> to describe this choice. The γ may <u>not</u> be C_p/C_v, but here it will be. Solving for the first two conditions gives:

$$J^2 = \frac{P_0 - P_1}{\rho_0 - \rho_1} \, ,$$

also called the shock adiabat, and finally solving for $\frac{\rho_1}{\rho_0}$ in terms of γ and P_1/P_0 gives:

$$\frac{\rho_1}{\rho_0} = \frac{v_0}{v_1} = \frac{(\gamma+1)P_1 + (\gamma-1)P_0}{(\gamma-1)P_1 + (\gamma+1)P_0}$$
$$\xrightarrow[P_1 \gg P_0]{} \frac{\gamma+1}{\gamma-1} \, .$$

(The problem is just one of algebra but I'll ask that <u>you</u> fill in the steps here; just notice that

$$\frac{1}{2}(v_1^2 - v_0^2) = \frac{1}{2}(v_1 - v_0)(v_1 + v_0)$$
$$= \frac{1}{2} v_1^2 \left(1 - \frac{\rho_1}{\rho_0}\right)\left(1 + \frac{\rho_1}{\rho_0}\right)$$

and:

$$\rho_1 v_1^2 - \rho_0 v_0^2 = \rho_1 v_1^2 \left(1 - \frac{\rho_0}{\rho_1} \cdot \frac{\rho_1^2}{\rho_0^2}\right) = \rho_1 v_1^2 \left(1 - \frac{\rho_1}{\rho_0}\right) = P_0 - P_1$$

which makes the algebra more clear <u>and</u> perhaps even more physical.)

The most important point is that <u>in the frame of the shock</u>, $v_0 = v_\Sigma$ if the gas to the right is at rest in the laboratory frame, so you know that the gas moves at a speed less than that of the front; in the laboratory frame it moves at $\frac{2}{\gamma+1} v_\Sigma$. So this is an important diagnostic of the compression, since $\frac{\rho_1}{\rho_0} \to \frac{\gamma+1}{\gamma-1}$. For a perfect monoatomic gas, $\gamma = 5/3$ so the compression ratio is 4; for a diatomic or more complicated species, where $\gamma < 5/3$, this is larger (more internal modes for the enthalpy means you have sinks — and the medium is more compressible).

If we continue ~~with~~ with this line of development, we can ask how it's possible for a shock to exist? You would expect that any compression should expand and "communicate" its presence to the gas ahead of it, right?

We've just looked at the condition for a discontinuity, where $\Delta P/P \sim P$. This is <u>very</u> nonlinear. But what if, instead, we take the fluid equations and ask for only <u>small</u> changes to propagate. Say:

$$\rho \to \rho_0 + \delta\rho$$
$$P \to P_0 + \delta P$$
$$v \to v_0 + \delta v$$

<u>and</u> assume $v_0 = 0$. Again I'm going to be writing $\underset{\sim}{V} = v\hat{x}$ using $V_j = \langle v_j \rangle$. The time dependent equations are:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho V_j) = 0 \to \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\rho v = 0$$

$$\rho\left(\frac{\partial V_j}{\partial t} + V_k \frac{\partial V_j}{\partial x_k}\right) + \frac{\partial P}{\partial x_j} = 0 \to \rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) + \frac{\partial P}{\partial x} = 0$$

and, for now, we'll ignore the energy equation by assuming constant entropy. Then substituting the ~~ll~~ perturbations, and assuming P_0 and ρ_0 are constant, all terms such as $V_k \partial_k V_j \to 0$ since they're nonlinear:

$$\frac{\partial}{\partial t}\delta\rho + \rho_0 \frac{\partial v}{\partial x} = 0$$

$$\rho_0 \frac{\partial v}{\partial t} + \frac{\partial P}{\partial x} = \rho_0 \frac{\partial v}{\partial t} + \left(\frac{\partial P}{\partial \rho}\right)_0 \frac{\partial \delta\rho}{\partial x} = 0$$

This pair of <u>hyperbolic</u> equations becomes:

$$\frac{\partial^2}{\partial t^2}\delta\rho = \left(\frac{\partial P}{\partial \rho}\right)\frac{\partial^2 \delta\rho}{\partial x^2} \to \left(\frac{\partial^2}{\partial t^2} - c_s^2 \frac{\partial^2}{\partial x^2}\right)\delta\rho = 0$$

so you see that $\left(\frac{\partial P}{\partial \rho}\right)_0$ has the dimensions of a speed which we will write as:

$$\left(\frac{\partial P}{\partial \rho}\right)_0 = c_s^2,$$

called the <u>sound speed</u> (only for a polytropic ideal gas is $c_s^2 = \frac{\gamma P_0}{\rho_0}$, this is a specific choice!). Thus, you have the condition we've asked about:

<u>NB!</u> If $v_\Sigma > c_s$ the gas cannot communicate the motion of Σ ahead of the front by simple compression — a sound wave — and the front will overtake the gas. This <u>defines</u> the Mach number, $\mathcal{M} \equiv v_\Sigma / c_{s_0}$.

The conditions (C_1, C_2, C_3) allow you to solve for the motion of Σ independent of time, so they hold at all moments in time at Σ.

<u>NB</u> You may have noticed, though, that we ignored the components <u>along</u> Σ, those that are transverse to the front. Here $\underset{\sim}{v} \times \hat{n}$ is simply identical. The situation is the same as the one you'll recall for ~~motion~~ wave propagation (or just the fields) in electromagnetic problems. The shock slows along $\hat{n}$ but the component $\underset{\sim}{V} \times \hat{n}$ doesn't change so the shock refracts the motion away from the normal. I'll ask you to think about what this tells you about the sound speed in the post-shocked gas, OK?