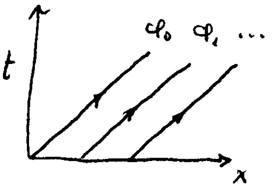
For a simple wave, for which c_s is constant, the solution for the amplitude:

$$\begin{pmatrix}\delta\rho \\ \delta v\end{pmatrix} \to a f_1(\underset{\sim}{x} + c_s t \hat{k}) + b f_2(\underset{\sim}{x} - c_s t \hat{k})$$

or, in words, there are "lines" along $(\underset{\sim}{x}, t)$ that move as a plane wave (or any other geometry) with constant phase and the propagation speed is the same as the phase speed of the wave. Another way to say this is that since c_s is constant, nothing about the wave changes. Of course, the addition of viscosity (collisions) damps the amplitude but that is not the same as changing its form.

These lines, in the (x,t) plane (I'll concentrate on 1D for now), form the solution of a Cauchy problem: only the initial conditions matter. If you know the solution at $t=0$, the lines of constant phase are just determined as straight lines with slope c_s (think here of EM waves if it helps, propagating in a vacuum).



But what if the amplitudes are not small so the equations don't linearize?

Again write:

$$\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0,$$

so rewrite this as:

$$\frac{\partial\rho}{\partial t} + \left(v + \frac{\partial v}{\partial\rho}\rho\right)\frac{\partial\rho}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + \left(v + \frac{1}{\rho}\frac{\partial P}{\partial\rho}\cdot\frac{\partial\rho}{\partial v}\right)\frac{\partial v}{\partial x} = \frac{\partial v}{\partial t} + \left(v + \frac{c_s^2}{\rho}\frac{\partial\rho}{\partial v}\right)\frac{\partial v}{\partial x} = 0,$$

Now you see we have the equations in a new form:

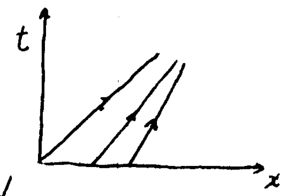
$$\frac{\partial\rho}{\partial t} + U_0\frac{\partial\rho}{\partial x} = \frac{\partial v}{\partial t} + V_1\frac{\partial v}{\partial x} = 0.$$

For the case $U_0 = U_1$, we then have the condition:

$$v + \frac{c_s^2}{\rho}\frac{d\rho}{dv} = v + \frac{dv}{d\rho}\rho$$

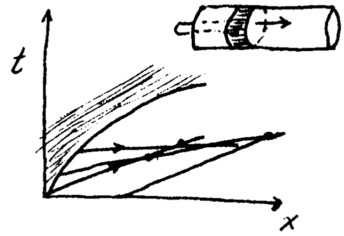
so the lines along which the solution from $t=0$ move will be given by

$$\left(\frac{dv}{d\rho}\right)^2 - \frac{c_s^2}{\rho^2} = 0 \to dv \pm \frac{c_s\, d\rho}{\rho} = 0$$



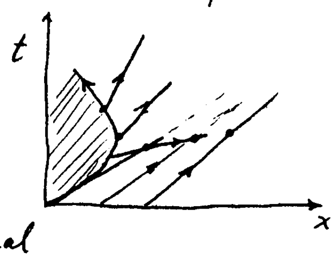
Depending on v at $t=0$, it's possible that something strange may happen, as I've drawn at right: the lines of constant phase may intersect! No physically acceptable solution can be double-valued, two separate values of φ can't exist at the same (x,t) so you see that the wave front changes with time. In this case, one moves faster than the other. The condition is expressed by:

$$v - v_0 = \pm\int\frac{c_s\, d\rho}{\rho}$$



which you can see from the illustration would be the case for a piston moving into a cylinder of gas at $v > c_{s,0}$, the sound speed for the unperturbed gas ahead of the piston (in English, the upstream gas). This is a shock and the location of this shock is given by the points at which these lines, ~~...~~ emanating from the piston, intersect downstream.

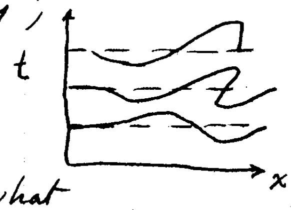
The lines are called <u>characteristics</u>; for the piston there's nothing coming from the region to the left of the piston and I've drawn the case of a piston moving <u>into</u> the gas at an accelerating rate.



If we imagine instead the case of a piston first sent into and then withdrawn from the gas, you see first <u>compression</u> and then <u>rarefaction</u> (lower density) resulting from the flow. Withdrawing the piston creates a set of lines that nowhere intersect (show this) so the signal propogates into the gas only at the local sound speed. Now if we have:

$$c_s^2 = \frac{\partial P}{\partial \rho} = \frac{\gamma P}{\rho} = \frac{\gamma P_0}{\rho_0^\gamma} \rho^{\gamma-1}$$

then if the density is a function of phase and c_s is a function of ρ, then for $\gamma > 1$ the high density regions must overtake those at low density, and as I've put in the cartoon, the wave forms a shock followed by a rarefaction. This may all seem very far from astrophysics, but it's essential: cosmic fluids are always compressible and strong waves are the norm in many cases - what starts out as a sound wave, if <u>not</u> damped (so $\delta\rho/\rho \to 0$) can become a shock (if $\delta\rho/\rho \to 1$). An example is a wave in an isothermal atmosphere - since



$$\rho \sim e^{-x/H_0}$$

where H_0 is the characteristic length scale for the atmospheric density, if the wave doesn't damp, some $x > 0$ exists for which $\delta\rho/\rho$ becomes large, whatever $\delta\rho$ is. Thus, in the ideal case, the wave <u>must</u> turn into a shock and because of now irreversible processes, the shock damps and locally <u>heats</u> the gas. <u>Note</u> that the essential feature of a characteristic is that it guarantees that once you know the physical state of the gas at $x(t=0)$, you know at all future times - until intersection - what position $x(t)$ has that <u>same</u> value of the parameters. So for $\rho(x,0)$, you know $\rho(x_\pm, t) = \rho(x_0, t)$ and to find the flow values, you re-order the set $\rho(x_\pm, t)$. This is how modern fluid codes work - called <u>Godonov method</u> codes: they construct a <u>grid of the characteristics</u> and solve the forward propogation problem by looking at the times Δt at which two characteristics, one from $x_0^+(t-\Delta t)$ and another from $x_1^-(t-\Delta t)$ intersect at t; that is for $x_1 > x_0$, using the $x_0 + \int \frac{d\rho\, c_s}{\rho}$ curve and the other using the $x_1 - \int \frac{c_s d\rho}{\rho}$ curve, you solve the algebraic equations.

<u>NB</u>: The shock jump conditions were those we derived as flux conservation laws, called the <u>Rankine-Hugoniot</u> relations. Notice that they hold for inviscid flows and are not restricted to ideal, or polytropic, gases. These provide the boundary conditions for the next problem we will discuss, the case of the strongest shocks - blast waves. But I should mention that photons - electromagnetic waves - can also form something like a shock. Consider a plane wave traveling into a medium of variable index of refraction. Then there is, clearly, a phase speed defined by $c(n)$, the waves don't only disperse but also bend. The intersection is a <u>caustic</u>, or a focus (for a good imaging system the intersection is at a point, not on a surface, but for a poor imager, this is more complicated — look at a light imaged by a glass of wine or water, for example, to get an idea of what a shock looks like).

From the fluid equations, you see that two quantities, (x, t), govern the evolution of all physical quantities. In other terms, all solutions to the equations of conservation are <u>evolution equations</u>. But if any quantity is strictly conserved (e.g. initial energy) then the other variables must change in a constrained way. Provided there are no special length scales, or time scales, to the evolution, then from dimensional analysis,

$$[E] = l^2 t^{-2} m \rightarrow \left[\frac{E}{m}\right]^{1/2} t \sim l$$

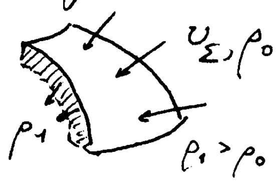
is now an example of a scaling relation. Note, however, that for a flow, there's a relation between m and ρ:

$$[m] = \rho l^3$$

so if the density of the medium remains constant,

$$\left[\frac{E}{m}\right] \rightarrow \left(\frac{E}{\rho_0}\right)^{1/2} l^{-3/2} t \sim l \rightarrow l \sim \left(\frac{E}{\rho_0}\right)^{1/5} t^{2/5}$$

Now imagine l is actually the surface of a sphere expanding into a region of density ρ_0 = constant, to be specific we'll say this is the result of an explosion in a body whose size was <u>much</u> smaller than $R(t)$ is at any time $t > 0$. If this is a supersonic expansion, gas flows through the front with the velocity $v_\Sigma = dR(t)/dt$ and compresses at the expense of the energy of the shock. Thus, without additional energy sources (E_0 = constant) the energy density must fall and the shock, in doing work on the background, must decrease in speed to maintain constant energy. It then follows that <u>within</u> the volume defined by $R(t)$ the density, etc. depends on the distance <u>toward</u> the center. <u>But</u> if the energy is constant, the change in these volumetric terms must always be the same relative to the physical size of the front – that is, we can write all physical quantities $Q(r,t)$ as:



$$Q(\underset{\sim}{x}, t) \rightarrow \tilde{Q}\left(\xi(\underset{\sim}{x}, t)\right), \quad \xi \equiv \frac{r}{R(t)}$$

where $\tilde{Q}$ is a <u>transformed</u> form of Q, <u>scaled by the dimensions</u>. For example:

$$\rho(r,t) = m l^{-3} \tilde{\rho}(\xi) \equiv m l^{-3} D(\xi)$$
$$v(r,t) = l t^{-1} \tilde{v}(\xi) \equiv l t^{-1} V(\xi)$$
$$P(r,t) = m l^{-1} t^{-2} \tilde{P}(\xi) \equiv m l^{-1} t^{-2} \Pi(\xi)$$

(note this has the same dimensions as energy density.)

This means we can write

$$\xi = \frac{r}{R(t)} \rightarrow \frac{r}{\left(\frac{E_0}{\rho_0}\right)^{1/5} t^{2/5}}$$

and transform from $(r,t) \rightarrow \xi$, so:

$$\left.\begin{aligned} \frac{\partial}{\partial r} &= \frac{d}{d\xi} \cdot \frac{\partial \xi}{\partial r} \\ \frac{\partial}{\partial t} &= \frac{d}{d\xi} \cdot \frac{\partial \xi}{\partial t} \end{aligned}\right\} \text{in general} \left\{ \begin{aligned} \frac{\partial}{\partial r} &= \frac{1}{R(t)} \frac{d}{d\xi} = \frac{\xi}{r}\frac{d}{d\xi} \\ \frac{\partial}{\partial t} &= -\frac{r}{R^2(t)} \frac{dR}{dt} \frac{d}{d\xi} \\ &= -\xi\left(\frac{1}{R}\frac{dR}{dt}\right)\frac{d}{d\xi} \end{aligned}\right.$$

Another way of looking at this is that you are able to stretch the relative coordinate, in real space, by scaling to how far the front Σ has gone. So, since R is only a function of time and must depend only on an exponent, i.e. $R \sim t^n$, $n > 0$, then

$$\frac{\dot{R}}{R} = n t^{-1}$$

so, for example,

$$\frac{\partial v}{\partial t} = \frac{\partial \xi}{\partial t}\frac{d}{d\xi}\left[\frac{r}{t}V(\xi)\right] = -\xi\frac{n}{t}\cdot\frac{r}{t}\frac{dV}{d\xi} + V(\xi)\frac{\partial}{\partial t}\left(\frac{r}{t}\right)$$

$$= -\frac{nr}{t^2}\xi\frac{dV}{d\xi} - \frac{r}{t^2}V(\xi)$$

$$\frac{\partial v}{\partial r} = \frac{\partial \xi}{\partial r}\frac{d}{d\xi}\left[\frac{r}{t}V(\xi)\right] = \xi\cdot\frac{1}{t}\frac{dV}{d\xi} + \frac{1}{t}V$$

$$v\frac{\partial v}{\partial r} = \frac{1}{t}\cdot\frac{r}{t}V\left[\xi\frac{dV}{d\xi} + V\right],$$

$$\frac{dv}{dt} = \left[\frac{\partial}{\partial t} + v\frac{\partial}{\partial r}\right]v \Rightarrow \frac{r}{t^2}V\left\{\xi\frac{dV}{d\xi} + V\right\} - \frac{r}{t^2}\left\{n\xi\frac{dV}{d\xi} + V\right\}$$

$$= \frac{r}{t^2}\left\{(V-n)\frac{dV}{d\xi} + V(V-1)\right\},$$

you've reduced the acceleration now to a scaled term — notice that rt^{-2} multiplies the now dimensionless term in $\{\cdots\}$. Multiplying by ρ would add a term $D(\xi)r^{-3}$ to consideration. (You know all terms come in identically in dimension so, for example, if the acceleration is from gravitation, you would have $-\frac{GM}{r^2}$, a bit more complicated if M is external; we'll return to this at the end of the course!).

Of course, it's natural to now ask : what does this all come to? Let's continue :

$$\frac{\partial}{\partial t}\rho = -\frac{\xi}{t} r^{-3}\frac{dD}{d\xi}$$

$$v\frac{\partial}{\partial r} r^{-3}D = \left(-\frac{3}{r^4}D + \frac{\xi}{r}\cdot\frac{1}{r^3}\frac{dD}{d\xi}\right)\frac{r}{t}V$$

$$\rho\frac{\partial}{\partial r}V\frac{r}{t} = r^{-3}D\cdot\frac{1}{t}V + \frac{\xi}{r}\cdot\frac{1}{r^3}\cdot\frac{D}{t}\frac{dV}{d\xi}$$

$$\rightarrow \quad -\xi\frac{dD}{d\xi}V + DV + \xi\frac{dV}{d\xi}D - 3DV + \xi V\frac{dD}{d\xi} = 0$$

which is the dimensionless form of the continuity equation. So again, once you know how the front Σ moves, you know the entire structure of the front. How? Because if we scale all variables to their upstream values, the Rankine-Hugoniot conditions give us $D(1)$, $\Pi(1)$, $V(1)$ and, therefore, the conditions for our hyperbolic system. Once done, it's done forever (me credi adesso?).

Return now to :

$$R(t) = \left(\frac{E_0}{\rho_0}\right)^{1/5} t^{2/5};$$

taking $\dot{R}$ we see that :

$$V_\Sigma(t) = \frac{2}{5}\frac{R(t)}{t} \sim R^{-1/2}\,E_0^{1/2}$$

and since the downstream (post-shocked) gas moves at $\frac{3}{4}V_\Sigma$, we have :

$$v = \frac{3}{10}\frac{R}{t}.$$

Thus, if you measure an emission line velocity from the gas and you know the size (radius) of Σ, you know the expansion age. Clearly, $\dot{R}\sim t^{-3/5}$. So there is also a speed - $M=1$, when $V_\Sigma \leq c_{s,0}$. Then :

$$c_s \sim \frac{R_*}{t_*} \qquad (R_* \equiv R(t_*) \text{ at some time } t_*)$$

$$\longrightarrow \quad R(t_*) \sim E_0^{1/3}.$$

This can also be found by asking when $P_\Sigma = P_0$, but since you know that P is the same as an energy density :

$$P_0 = \frac{E_0}{R(t_*)^3} \rightarrow R(t_*) \sim E_0^{1/3},$$

where we now substitute (E_0/ρ_0) for E_0. To give an example, the Crab nebula has $R\sim 1\,\mathrm{pc}$ and $v\sim 10^3\,\mathrm{km\,s^{-1}}$ so it has an expansion age of order centuries (I'm just using rough numbers!). Now this means we have, for $\rho_0 \sim 1\,\mathrm{cm^{-3}}\cdot 10^{-24}\,g$, $E_0/\rho_0 \rightarrow [(3\times10^{18})^{5/2}\,10^{-10}]^2\rho_0 \sim E_0 \gtrsim 10^{48}\,\mathrm{erg}$, a huge number! (for $L_\odot \sim 4\times10^{33}\,\mathrm{erg\,s^{-1}}$ this is the energy produced by the Sun in 10^{10} yrs). Since $E_0 \gg L_\odot t_{exp}$ for an expansion time t_{exp}, we can conclude that this input was dynamical on a <u>very</u> short timescale, an explosion. In fact, given the gravitational self energy of the Sun, $\sim 10^{53}\,\mathrm{erg}$ $\left(\frac{3}{10}\frac{GM^2}{R}\right)$, you see that $> 10^{-5}E_\odot$ went into the kinetic energy of the explosion!!

<u>NB</u> $\Rightarrow$ This is a <u>supernova</u>, the simplest estimate of the order of magnitude of the event, and it is very much an underestimate.

The similarity approach yields many useful scaling laws when $t \gg \Delta t$, the timescale for the event, or when $R(t) \gg R(0)$:

1. Constant momentum: conserved quantities are ρ_0 and p_0, the initial momentum,
$$R(t) \sim \left(\frac{p_0}{\rho_0}\right)^{1/4} t^{1/4}$$

2. Constant mechanical luminosity: conserved quantities are $\underbrace{\dot{m} v_\infty^2}_{(L_w)}$ and ρ_0 so,
$$R(t) \sim \left(\frac{L_w}{\rho_0}\right)^{1/5} t^{3/5} \quad \text{(we'll compute } L_w \text{ soon)}$$

3. Constant pressure: conserved quantities are $P_0 = \rho \underbrace{v^2}_{c_s}$ and ρ_0,
$$R(t) \sim c_s t,$$

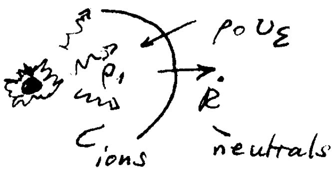
This last solution is referred to in the fluids literature as a stalled shock since it expands at essentially constant speed. Form (1) is the late time evolution of an adiabatic (Sedov-Taylor) solution, form (2) is expected for the "bubble" from an association or when a strong stellar wind is present. Remember, these are merely asymptotic solutions to the flow equations. They have the feature that
$$\frac{R(t)}{\dot{R}(t)} \sim \frac{R(t)\, t}{n\, R(t)} \sim \frac{t}{n} \quad \forall\, t$$

so there is no characteristic expansion timescale. If, however, τ is a timescale that's appropriate for some process (i.e. radiative cooling) then when $t/n \sim \tau$, we cannot (or may not be able to) use the conserved quantity we'd chosen at the start.

NB Not all scaling laws require time. For instance, a shell expanding with constant mass follows the law that $\Delta M \sim \rho_0 r^3 \to \rho(r) \sim r^{-3}$ and therefore $v \sim r$ to insure that $\rho v r^2$ is constant (the case of a nova shell, for instance, before mass is loaded into the flow, $\rho > \rho_0$).

NB Next semester, in cosmology, you'll see the outstanding example of a scaling law, the Friedmann-Robertson-Walker equations; these describe the evolution of the radius of curvature of the Universe assuming a constant curvature for spacetime.

For another solution, constant ionization, the calculation must be a bit more precise. This is the case of an HII region. Consider a star of some constant luminosity, L, of which L_{UV} is the ionizing part. Then the UV continuum will ionize the surrounding - presumed to be neutral - gas by an equation of the ionization rate, β_{UV}, and the recombination rate, α:



$$\beta_{UV}\, n_0 = \alpha\, n_1^2 R^3$$

and we'll use ρ_0 and ρ_1 indiscriminently since only a constant factor of mass is missing. Then,
$$\rho_1 = \left(\frac{\beta_{UV}}{\alpha}\rho_0\right)^{1/2} R^{-3/2}.$$

OK, now since the ionized gas has a very high pressure relative to the upstream neutral material, $P_0 \ll P_1 = \rho_1 c_s^2$, where c_s is the post-shocked value of the sound speed. Then writing:

gives the approximate condition:

$$\rho_0 \dot{R}^2 \simeq \rho_1 c_{S,1}^2 ; \qquad (c_{S,1}^2 \gg c_{S,0} ; \ \rho_0 c_{S,0}^2 \ll \rho_0 \dot{R}^2)$$

substituting for ρ_1 gives

$$R^{3/2} \dot{R}^2 \simeq \left(\frac{\beta_{UV}}{\alpha}\right) \rho_0^{-1/2} \equiv A$$

so we obtain

$$R^{3/4} \dot{R} = A^{1/2} \rightarrow R(t) \sim A^{1/2} t^{4/7} \sim \left(\frac{\beta_{UV}}{\alpha \rho_0^{1/2}}\right)^{1/2} t^{4/7}$$

at late times in the expansion. In fact, there are two timescales here, due to α and β_{UV}. But they enter as a simple ratio so the two dimensionally cancel each other. Note that for the continuity equation - the mass flux in the Rankine - Hugoniot conditions - we use instead a replacement. We say that the influx of UV photons is balanced by the inflow of mass in the frame of the shock.

NB I'm ignoring the geometrical factors here and some numerical factors have been dropped but never mind!

What about the energy equation? To do this correctly, that is to find the compression ratio properly, we require including the ionization potential ϵ since the state of the gas changes from neutral to ionized we must use a complete expression for the energy; what happens if we're also dissociating molecules?) $\Rightarrow$ The rest of this problem is left for you to examine, but I'll make one remark: if you include the effects of a wind, the problem becomes one of computing how this wind expands within the initial HII region and then breaks out since, unlike the ionization zone, it's not static. Matter "flowing through" the shock (recall that you're in the Σ' frame) is ionized by this radiation field.