To first, again from class, derive the Navier-Stokes equation, we note that the momentum flux from coupling together regions that are shearing adds a term to the stress tensor of the form:

$$T_{ij} = \rho v_i v_j + P\delta_{ij} - \eta \frac{\partial v_i}{\partial x_j}$$

where now $\eta = \rho\nu$, with ν being the so-called kinematic viscosity. Really, we should use:

$$\sigma_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right),$$

the strain, since you already know that $T_{ij} = T_{ji}$ (there are only six (6) independent components of T_{ij} by this symmetry). If we take ρ = constant and η = constant, then because ρv_i is divergenceless we can ignore the $\partial v_j/\partial x_i$ term (since $\partial_j T_{ij} = \frac{1}{2}[\partial_j^2 v_i + \partial_i(\partial_j v_j)] \rightarrow \partial_j T_{ij} = \partial_j^2 v_i$ in this case). Then:

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 v_i}{\partial x_j \partial x_j}.$$

This is the Navier-Stokes equation (well, really the NSE is without $\partial_i P$ but since this is required for our purposes, let's leave it in place). Now we have a remarkable scaling property. Choose L and U to be characteristic scales of length and velocity. Then:

$$U\frac{\partial \tilde{v}_i}{\partial t} + \frac{U^2}{L}\tilde{v}_j\frac{\partial \tilde{v}_i}{\partial \tilde{x}_j} = -\frac{1}{\rho}\frac{\partial P}{\partial \tilde{x}_i}\cdot\frac{1}{L} + \nu\frac{U}{L^2}\frac{\partial^2 \tilde{v}_i}{\partial \tilde{x}_j \partial \tilde{x}_j}$$

so since the dimensions of ν are $[\nu] = L^2 t^{-1}$, $t \rightarrow \frac{L^2}{\nu}\tilde{t}$ and thus:

$$\frac{\partial \tilde{v}_i}{\partial \tilde{t}} + Re \cdot \tilde{v}_j \frac{\partial \tilde{v}_i}{\partial \tilde{x}_j} = \frac{L}{U\nu}\cdot\frac{1}{\rho}\frac{\partial P}{\partial \tilde{x}_i} + \frac{\partial^2 \tilde{v}_i}{\partial \tilde{x}_j \partial \tilde{x}_j}$$

is dimensionless. But we now have a pure number:

$$Re \equiv \frac{UL}{\nu} = (\text{Reynolds number}) = \frac{(\text{inertial term})}{(\text{viscous force})}$$

that serves to scale our solutions. That is, all solutions for $\tilde{v}_i$ with the same Re are identical. We also see that we can form a similarity variable:

$$\xi = r t^{-1/2} \quad (\text{since } [\nu] = L^2 t^{-1})$$

that renders the NSE in a form like the hyperbolic system we've been using for inviscid flows (I'll leave it to you to explore what the resultant is for this, OK?). Just as a note, remember that you must use $v_i = \frac{r}{t}V(\xi)$.

We have one more feature to examine for this equation before proceeding. There's an important vector identity you should recall:

$$\nabla^2 \underset{\sim}{A} = \nabla(\nabla\cdot\underset{\sim}{A}) - \nabla\times(\nabla\times\underset{\sim}{A})$$

along with:

$$-A\times(\nabla\times B) = A\cdot\nabla B - \underset{\sim}{A}\,\nabla\cdot B$$

so if $\underset{\sim}{A} \Rightarrow \rho\underset{\sim}{v}$ and $\underset{\sim}{B} \Rightarrow \underset{\sim}{v}$ then:

$$\rho v_j \frac{\partial v_i}{\partial x_j} - \rho v_j \frac{\partial v_j}{\partial x_i}$$

$$-\rho\underset{\sim}{v}\times(\nabla\times\underset{\sim}{v}) = \rho\underset{\sim}{v}\cdot\nabla\underset{\sim}{v} - \rho\underset{\sim}{v}\cdot(\nabla\underset{\sim}{v})$$

We will introduce a new definition, the vorticity:

$$\underset{\sim}{\omega} \equiv \nabla\times\underset{\sim}{v}$$ (you've seen this before in electrodynamics, recall $\underset{\sim}{B} = \nabla\times\underset{\sim}{A}$ from the vector potential)

so now from the equations of motion:

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right) \rightarrow \rho\left(\frac{\partial v_i}{\partial t} + v_i\frac{\partial v_i}{\partial x_j} - \epsilon_{ijk}\, v_j \omega_k\right)$$

where ε_{ijk} is the permutation symbol,

$$\varepsilon_{ijk} v_j \omega_k = (\underset{\sim}{v} \times \underset{\sim}{\omega})_i$$

Then since $\nabla \times \nabla P = 0$ always, if $\nabla \times \frac{1}{\rho} \nabla P = \nabla \frac{1}{\rho} \times \nabla P = 0$ (I'll return to this soon), we have:

$$\nabla \times \left[\rho\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_j}{\partial x_i} - (\underset{\sim}{v} \times \underset{\sim}{\omega})_i\right)\right] = \eta \nabla^2 \omega_i$$

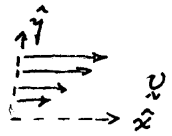
which looks very much like an equation you've seen before – the Lorentz force in electromagnetic theory. In other terms:

$$\rho \left(\frac{\partial \underset{\sim}{\omega}}{\partial t} + \nabla \times (\underset{\sim}{\omega} \times \underset{\sim}{v})\right) = \eta \nabla^2 \underset{\sim}{\omega} .$$

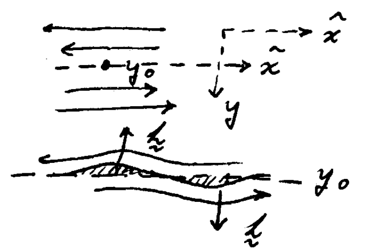
This nonlinear evolution equation for the vorticity is at the core of much of the modern theories of turbulence and directly relates to Re. If we have

$$[\omega] = U L^{-1} \rightarrow Re = [\omega] L^2/\nu$$

so in the limit of large vorticity, we have a medium that can transport the twisted lines of fluid from shear very quickly, before viscosity can smear them out. This means no streamlines exist in the fluid – they're twisted by $\underset{\sim}{\omega}$. In fact, this is the principal source for turbulent viscosity, the coupling of different parts of the fluid together by $\underset{\sim}{\omega}$ (unlike $\underset{\sim}{v}$ for a perfect fluid, you always have $\underset{\approx}{\omega}$ connecting different regions because it depends on a spatial derivative). Further, since $\nabla \cdot \underset{\sim}{\omega} = 0$ by definition, $\nabla^2 \underset{\sim}{\omega} = -\nabla \times (\nabla \times \underset{\sim}{\omega})$ so the diffusive term can be thought of as a further tangling of the fluid.



Why all this talk about vorticity? Recall my discussion of a shear flow. If we imagine $\underset{\sim}{v} = v_x(y)\hat{x}$, then $\partial v_x/\partial y \neq 0$ and therefore $\nabla \times \underset{\sim}{v} = \underset{\sim}{\omega} \neq 0$ at any point. So choose some y_0 as a reference. Then at $y > y_0$, $v_x(y) > v_x(y_0)$ and $v_x(y < y_0) < v_x(y_0)$. Now imagine you perturb the boundary along y_0. Flow is diverted, and therefore accelerated. As we've discussed in class, in the absence of ν, this produces oppositely directed lift, $\underset{\sim}{L}$, of equal and opposite magnitude:



$$\rho \ddot{\delta y} = \rho \underset{\sim}{L} \rightarrow \omega_0^2 \, \delta y = L_y ; \qquad \begin{array}{l}\omega_0 = \text{frequency} \\ \omega = \text{vorticity}\end{array}$$

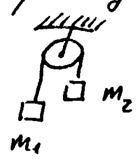
by Bernoulli's theorem,

$$L_y \sim \rho \frac{v_x}{L} |\Delta v_x| \qquad (k \sim L^{-1})$$

so since $\Delta v_x / L \sim \omega$, then $\omega_0^2 \sim |\Delta v_x| k$; in this case, any shear is unconditionally unstable, and the rate of growth is proportional to $|\Delta v_x|^{1/2}$, and is fastest for the smallest wavelengths (k large).

Now look at the case $\nu \neq 0$. Then the acceleration is $\nu k^2 \omega$ for the vorticity so on a timescale of an eddy turnover (the same as ω^{-1} or, for the shear instability, the same as ω_0^{-1}), at some length the viscosity can smooth out the perturbations. In fact, only for a length L larger than $(\nu t)^{1/2} \sim (\nu \omega^{-1})^{1/2}$ will the instability be able to develop. So the Reynolds number appears directly! This instability is the most asic for <u>any</u> fluid, called in the inviscid ($\nu = 0$) case the Kelvin-Helmholtz instability (first realized around 1857!), and is certainly the most basic for a sheared fluid. It is also, as we'll soon see, damped in the case where angular momentum is transported in the flow (planar flow doesn't involve any net transport of angular momentum <u>but</u> circulation does).

Now we'll pass to the next case, two superimposed fluids under gravity. To use the example from lecture, consider an <u>Atwood machine</u> with masses m_1 and m_2. Then for z being the displacement, the Lagrangian is:

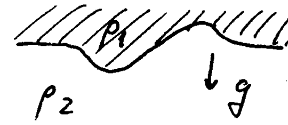


$$\mathcal{L} = \frac{1}{2}m_1\dot{z}_1^2 + \frac{1}{2}m_2\dot{z}_2^2 - m_1 g z_1 - m_2 g z_2\ ;$$

but $z_1 + z_2 = 0$ for displacements so

$$\mathcal{L} = \frac{1}{2}(m_1+m_2)\dot{z}_1^2 - (m_1 - m_2) g z_1$$

and therefore the system moves with an acceleration:



$$a = \frac{m_1 - m_2}{m_1 + m_2}\, g$$

depending only on $m_1 - m_2$. Now if $m_j = \rho_j V$, for a volume V, we can pass to the form $a = \omega_0^2 k^{-1}$ so that

$$\omega_0^2 = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\, g k.$$

If ρ_1 is on top of ρ_2, and $\rho_1 > \rho_2$, $\omega_0^2 > 0$ and the fluid is unstable – for incompressible fluids ρ_1 moving down releases more potential energy than is required to lift an equal <u>volume</u> of ρ_2. This is the <u>Rayleigh-Taylor</u> instability, and you see that unlike the Kelvin-Helmholtz, it has a stable and unstable configuration.

<u>NB</u> Now how would you combine these two instabilities together for a planar flow with shear but different densities across an interface?

So now we're ready to examine, again, convection. <u>Any</u> buoyant motion is identical to the Rayleigh-Taylor instability – the interchange of densities. It only makes life easier to use incompressibility but it isn't essential (since the volumes must be the same for the case of an ideal incompressible fluid, or when $\nabla\cdot\underset{\sim}{v} = 0$, you can use the energetic arguments in their simplest form).

<u>NB</u> If $\nabla\cdot\underset{\sim}{v} = 0$ (ρ = constant), then $\underset{\sim}{v} = \nabla\times\underset{\sim}{\Psi}$ for some potential function $\underset{\sim}{\Psi}$ (as usual, $\nabla\cdot(\nabla\times\underset{\sim}{\Psi}) = 0$ always) so that $\nabla\times\underset{\sim}{v} = \underset{\sim}{\omega} = -\nabla^2\underset{\sim}{\Psi}$; in this case, <u>only</u> vorticity matters and the interchange occurs in rolling motions of closed circulation. This is the case for non-turbulent buoyant motions, they must generate shear, so you already see that all fluids that are buoyantly unstable also experience the growing modes of a Kelvin-Helmholtz instability and on <u>some</u> length scale cannot remain inviscid! È <u>Carina</u>, no?

Now what happens when we look at a rotating system? We've included this in the derivation of the equations of motion by using a cylindrically symmetric coordinate system:

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} - \frac{v_\varphi^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{1}{\rho}\frac{\partial P}{\partial r} + a_r$$

$$v_\varphi\frac{\partial v_\varphi}{\partial \varphi} \quad \frac{1}{r}v_r\frac{\partial}{\partial r}(r v_\varphi) + v_z\frac{\partial v_\varphi}{\partial z} = -\frac{1}{\rho}\cdot\frac{1}{r}\frac{\partial P}{\partial \varphi} + a_\varphi$$

$$v_r\frac{\partial v_z}{\partial r} + \frac{1}{r}v_\varphi\frac{\partial v_z}{\partial \varphi} + v_z\frac{\partial v_z}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + a_z$$

Now for an axisymmetric system, you'll notice that $\partial_\varphi \to 0$ and $a_\varphi = 0$ so angular momentum is not only conserved but constant. Then $r v_\varphi = j$ is the specific angular momentum. For a symmetric system, v_z must vanish or it must change sign at $z=0$. So the first case we can examine is when $v_z = 0$. Then:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\rho v_r r = 0$$

gives the relation between the density and radius and

$$\frac{1}{\rho}\frac{\partial P}{\partial z} = -a_z$$

is the equation for the vertical structure. Notice that because angular momentum is constant, the rotation is assumed to occur on cylinders. I'll return to this equation soon. Notice now that:

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} - \frac{v_\varphi^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + a_r$$

can be written in the form of an effective potential. If $P = \rho c_s^2$ and, for the moment, we take v_r as a stationary flow, then we have again the wind equation:

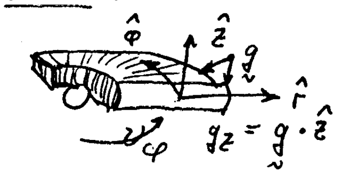
$$v_r\frac{\partial v}{\partial r} = -\frac{c_s^2}{\rho}\frac{\partial \rho}{\partial r} - \frac{\partial \Phi}{\partial r} + a_{rad} \qquad \Phi = \Phi_{grav} + \Phi_{cent},$$

That is, rotation hinders radial motion but since $-(\ln r + \ln v_r) = -\ln\rho$ then

$$\frac{1}{v}(v^2 - c_s^2)\frac{dv_r}{dr} = +\frac{c_s^2}{r} - \frac{\partial \Phi}{\partial r} + a_{rad} \; (=0 \text{ for now})$$

so $v_r = c_s^2$ when $\frac{\partial \Phi}{\partial r} = \frac{c_s^2}{r}$. This depends on v_φ but $v_\varphi r = j$ so that we see for $v_\varphi = v_\varphi(r)$ there will be a critical (sonic) point in the flow. If $a_{net,r}$ is due only to gravity, we have the equation for the opposite of a wind, an accretion flow. And because $v_\varphi \neq 0$, we have a disk!

When this sort of flow occurs, then we can make significant progress by taking



$$g_z \equiv a_z = g\cdot\hat{z} = -\frac{GM}{r^2}\frac{z}{r}$$

for a central potential with mass M. This is the so-called "thin disk" approximation in which we take $\cos\vartheta \equiv \hat{g}\cdot\hat{z} = z/r$. Then:

$$\frac{\partial \ln\rho}{\partial z} = -\frac{GM}{c_s^2 r^3}z$$

which means we have a gaussian exponential mass distribution with a thickness that depends on r. (Further restricting our approximation, if $v_\varphi^2 = \frac{GM}{r}$, the case for central field motion (called Keplerian motion) then:

$$\Omega^2 = \frac{GM}{r^3} \to \quad \left(\frac{v_\varphi}{c_s}\right) = \frac{r}{z_0} \to \quad z_0 = \frac{c_s}{v_\varphi}r;$$

the thickness z_0 grows with increasing r because the gravitational acceleration decreases with increasing distance from M.

Note: Aside

This approach is one that applies in very general cases. Let's return to the angular momentum equation:

$$\frac{1}{r}\rho v_r \frac{\partial}{\partial r}(r v_\varphi) = a_\varphi \rho$$

Then:

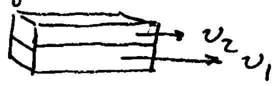
$$r\rho v_r = \frac{r^2 a_\varphi \rho}{\frac{\partial}{\partial r}(r v_\varphi)} \rightarrow \frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) = \frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left\{\frac{r^2 a_\varphi \rho}{\frac{\partial}{\partial r} r v_\varphi}\right\} = 0.$$

You see we have eliminated v_r from consideration! You still have v_φ in the equation but now take:

$$v_\varphi = \alpha r^{-1/2}, \quad \frac{\partial}{\partial r} r v_\varphi = \frac{1}{2}\alpha r^{-1/2},$$

leaving only the determination of a_φ. Here I'll digress and introduce the most general form of the equations of motion, the Navier-Stokes equation.

Suppose you have two regions that meet on a surface for which you have a difference in velocity – that is to say, momentum – with $v_1 > v_2$, for example. Then if there is friction, (2) slows (1) down while (1) speeds (2) up.



If the coefficient of friction is ν, then the force is:

$$F = \rho\eta\, \nabla\cdot\nabla v = \rho\nu \nabla^2 v$$

(look at the dimensions, this is a divergence of a momentum flux). Now in our case, $v = v_\varphi \hat{\varphi}$ so:

$$\nabla\cdot\nabla v = \hat{\nabla}_r \frac{\partial}{\partial r}(v_\varphi \hat{\varphi}) + \hat{\nabla}_\varphi \frac{1}{r}\frac{\partial}{\partial \varphi}(v_\varphi \hat{\varphi})$$

but the divergence acts only on the radial direction. Since $\frac{\partial \hat{\varphi}}{\partial \varphi} = -\hat{r}$ and $\frac{\partial \hat{\varphi}}{\partial r} = 0$,

$$\nabla^2 v_\varphi \hat{\varphi} = \nabla_r \left[\hat{r}\hat{\varphi}\frac{\partial v_\varphi}{\partial r} - \frac{v_\varphi}{r}\hat{r}\hat{\varphi}\right]$$

$$= \hat{\varphi}\left[r\frac{1}{r}\left(\frac{\partial v_\varphi}{\partial r}\right) - \frac{r v_\varphi}{r^2}\right] \quad \text{(integrate by parts)}$$

$$= \hat{\varphi}\, r\left[\frac{\partial}{\partial r}\left(\frac{v_\varphi}{r}\right)\right]$$