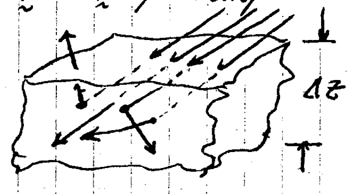
Now to continue with the discussion from today's class. For one case, scattering, we can find a solution that makes sense for S. This also introduces an essential, new idea:

> the transfer of radiation is the same as a probability evolution – in both scattering and absorption or emission the intensity is the probability we will see a photon (or anything else) from some direction.

This statement does not, however, mean the process is either linear or has no memory, there is feedback possible to the medium. The emission process, if it depends on the absorption of radiation in a coherent way, may have a "memory" so we can't always use the on-the-spot assumption. In other words, and I'll explain this in more detail later, the transfer may not be a Markov process (i.e. one that depends only on what happens at some $\underset{\sim}{x}$ with no correlation to $\underset{\sim}{x} \pm \Delta \underset{\sim}{x}$ for any $\underset{\sim}{x}$).

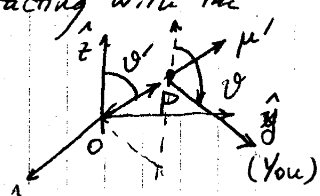
For the simplest single scattering process, we imagine some incident intensity, $I(\tau,\mu)$, at the top of a layer. Then we assert that the medium has some coefficient, σ, for scattering. Recall that this is a cross section, right? Then we ~~do~~ separate σ:



$$\sigma(\tau,\mu) \equiv \sigma_0(\tau)\, p(\mu,\mu') \quad \left[\text{or } \sigma_0(\tau)\, p(\hat{\Omega}\cdot\hat{\Omega}')\right]$$

where $p(\mu,\mu')$, which I called the phase function in class, is the probability that radiation coming from some direction $\mu' \to \mu$ after interacting with the medium. Thus:

$$\int d\mu\, d\mu'\, p(\mu,\mu') = 1$$



(NB: for calculation, look at the process at P.)

Now you see how scattering redirects the light:

$$I(\tau,\mu) = \sigma_0 \int I(\tau,\mu')\, p(\mu,\mu')\, d\mu'$$

and if $\nu \neq \nu(\mu,\mu')$ (remember, for Compton scattering this isn't true) then we can write the same expression explicitly as a function of frequency. Not knowing what to take for p, we can start with

$$p(\mu,\mu') = p_0 = \text{constant},$$

an isotropic scatterer. Take this to be a medium with $\kappa_\nu \to 0$. Then:

$$J(\tau) \equiv \frac{1}{2}\int_{-1}^{1} I(\tau,\mu')\, d\mu' = \frac{1}{2} M_0(\tau)$$
$$= \frac{1}{2}\left(I^+(\tau) + I^-(\tau)\right)$$

will now be called the mean intensity. Notice this is independent of μ, measured at any point in the medium. We can further define two more moments:

$$F(\tau) \equiv 2\int I(\tau,\mu)\,\mu\, d\mu$$

$$K(\tau) \equiv \frac{1}{2}\int_{-1}^{1} I(\tau,\mu)\,\mu^2\, d\mu.$$

(These definitions are conventional, and are independent of the process, but you'll need them for what follows and for looking at the literature.) We can now say that if we see light in any direction, μ, it's because it originated in some direction μ' so:

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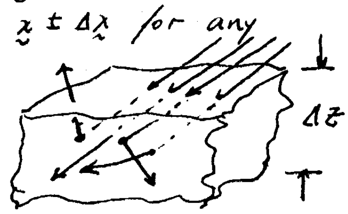

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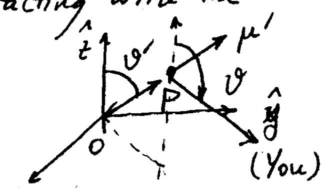

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(These definitions are conventional, and are independent of the process, but you'll need them for what follows and for looking at the literature.) We can now say that if we see light in any direction, μ, it's because it originated in some direction μ' so:

$$\mu \frac{dI}{d\tau} = I - J, \quad \text{where } \tau = \int \sigma \rho \, dz, \text{ the scattering depth}$$

and we therefore have a known source function:

$$S(\tau) = J(\tau);$$

~~and~~ this substitution is called, in the literature, the Milne-Schwarzschild problem. Actually, it's not all that realistic, but it also isn't too idealized.

Let's return to the formal solution:

$$\frac{d}{d\tau/\mu} e^{-\tau/\mu} I(\tau,\mu) = J(\tau) e^{-\tau/\mu}$$

with the condition that we now want the intensity at some depth τ:

$$e^{-\tau/\mu} I(\tau,\mu) = \int_\tau^\infty J(t) e^{-t/\mu} \frac{dt}{\mu} \to I(\tau,\mu) = \int_\tau^\infty J(\tau) e^{-(t-\tau)/\mu} \frac{dt}{\mu}$$

(it doesn't matter if we take J or S here, of course). Then:

$$J(\tau) = \frac{1}{2} \int_{-1}^{1} I \, d\mu = \int_\tau^\infty \frac{dt}{2} \int_{-1}^{1} \frac{d\mu}{\mu} e^{-(t-\tau)/\mu} J(t)$$

so calling $x = \frac{1}{\mu}$ then noting that $\frac{dx}{x} = -\frac{d\mu}{\mu}$ we have:

$$J(\tau) = \frac{1}{2} \int_\tau^\infty J(t) E_1(|t-\tau|) dt \qquad \text{(you should show this)}$$

having defined a new function, the <u>exponential integral</u>:

$$E_n(y) \equiv \int_1^\infty \frac{e^{-xy}}{x^n} dx.$$

There are many pretty features of this function, most of which are outlined in Hopf, Chandrasekhar, Busbridge, and Kourganoff. For now, I'll just note a few:

$$\frac{dE_n(y)}{dy} = -\int_1^\infty \frac{e^{-xy}}{x^{n-1}} dx = -E_{n-1}(y), \qquad n > 0$$

$$E_n(0) = \frac{1}{n-1}. \qquad n > 1$$

Notice that for $\tau \to 0$, the obvious result emerges that $I(0,\mu)$ is the Laplace transform of J (or S). But there's a much deeper ~~signific~~ significance to this relation. The random process we're invoking for the passage of radiation through the medium gives:

$$\begin{pmatrix} \text{Probability that light} \\ \text{emitted at } t \to \tau \end{pmatrix} = e^{-(t-\tau)/\mu}$$

which, in class, I called $p_{esc}(\tau,\mu)$ the <u>escape probability</u> for some direction μ, since defining $s = \tau/\mu$ we have $e^{-s} ds$, a single scattering probability. Integrating over $\mu d\mu$ gives the total probability:

$$p_{esc} = \frac{1 - e^{-\tau}}{\tau} \to \begin{cases} 1 & \tau \to 0 \\ 0 & \tau \to \infty \end{cases},$$

again regardless of the process. <u>And</u> you'll notice that:

$$J(\tau) = \frac{1}{2} \int_\tau^\infty S(t) K(t-\tau) \, dt \to S * K$$

for a kernel K (here E_1) is a convolution — averaging over angle is the ume as a convolution of the source function with a propagator. Here $E_1(x)$ is, in effect, that function so this formalism is also called the escape probability method. So to continue, for the moments, ~~d~~ having defined F and K, we have

$$\frac{dF}{d\tau} = 0, \quad F = \text{constant}.$$

This is fundamental: for a plane parallel medium, F - the flux - is constant; since $\tau \to \infty$ as $\mu \downarrow 0$, no radiation can "leak" out of the sides of the layer. For the next moment

$$\frac{dK}{d\tau} = \frac{1}{4}F \to \quad K = \frac{1}{4}F\tau + \text{const.}$$

You now see why we must close the moments; if we don't have some way of relating successive integrals, this process just continues forever. So if $K = \frac{1}{3}J$,

$$J(\tau) = \frac{3}{4}F\tau + \text{const.},$$

and we are finished, having found the source function in closed form. But what is the constant? Certainly it cannot be zero: at $\tau = 0$, the surface, $J(0) \to I^+(0)/2$ (assuming $I^-(0) \to 0$), no? So we must return to the formal solution and take the two necessary integrals (one for ~~Itop~~ $I(0,\mu)$, and the other for $J(0)$) to obtain:

$$J(\tau) = \frac{3}{4}F\left(\tau + \frac{2}{3}\right)$$

Aside We can jump to an essential point. Notice that $J(0) \neq 0$ has a very important implication, one I'll explain in more detail. If we identify J with $B(\tau)$, the integrated Planck function, then since $B = aT^4$, where a is the Stefan-Boltzmann constant and T is the temperature, using F as a reference temperature dependent quantity, σT_{eff}^4,

$$T^4(\tau) = \frac{3}{4}T_{eff}^4\left(\tau + \frac{2}{3}\right) \xrightarrow[\tau \to 0]{} \frac{1}{2}T_{eff}^4$$

The quantity T_{eff}, called the effective temperature, is defined using the luminosity and radius of the emitting body:

$$\mathcal{L} \equiv 4\pi R^2 \sigma T_{eff}^4 \qquad (\sigma \text{ is also a constant})$$

and we know this is constant through the layer. Then, really, $\tau = \frac{2}{3}$ is the optical depth at which $T(\tau) = T_{eff}$, not $\tau = 1$, although for $\mu = 1$ this second value is where $e^{-\tau}$ falls by e^{-1}.

NB This whole discussion is only correct for a plane parallel atmosphere. It fails completely for a spherical atmosphere and, we'll see, ~~on~~ for those cases for which the opacity, κ, has a strong dependence on wavelength.

Now back to our main discussion. In all results so far, you have assumed that κ, and all other quantities, are independent of ~~wave~~ frequency. This guarantees the result F = constant. But this is not true if we use κ_ν (and for this statement, you can substitute σ_ν). Only:

$$F = \int_0^\infty F_\nu \, d\nu$$

is constant. Thus, what any modification by the medium produces at one frequency is compensated by changes in all others to conserve flux. The result is that:

$$F_\nu = \frac{1}{3}\frac{dJ_\nu}{d\tau_\nu} \quad , \quad F = \frac{1}{3}\frac{dJ}{d\tau}$$

can be used to define an integrated opacity:

$$\frac{1}{\kappa_R} \equiv \frac{\int_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B_\nu}{\partial T}d\nu}{\int \frac{\partial B_\nu}{dT}d\nu} ,$$

called the Rosseland mean opacity. This quantity was first encountered in a different context, but the result is the same, it depends on the identification

$$J_\nu \to B_\nu(T)$$

and the use of:

$$\frac{dB_\nu}{d\tau_\nu} = \frac{1}{\kappa_\nu\rho}\frac{dB_\nu}{dz} = \frac{1}{\kappa_\nu\rho}\frac{\partial B_\nu(T)}{\partial T}\frac{dT}{dz}$$

(I'm only using $\frac{\partial}{\partial T}$ to emphasize that the derivative is only over T, not ν). Since you know the formal definition of $B_\nu(T)$, the derivative is also known.

NB This is a definition for κ_R, the physical justification for which will have to wait until we discuss the atmosphere problem in more detail.

The advantage of κ_R is its independence of frequency, precisely what we were looking for. If κ, or σ, is independent of ν, then $\kappa_R = \kappa$, of course. You'll notice that κ_R is weighted most heavily to the peak of B_ν, so depending on the composition, κ_R will have different dependences on T, hence we have an important result: the scaling law, $d\tau_\nu = \left(\frac{\kappa_\nu}{\kappa_R}\right)d\tau_R$ depends on the composition. The depth scale is assumed to be given by κ_R (or any other scaling) but this is only a reference. The "absolute depth" is meaningless. For instance, for a star, the atmosphere is matched to the interior at some τ_R and the emergent spectrum is computed using τ_ν. The same may be true for a planet, but not one with a solid surface for which we can use altitude rather than optical depth.

You'll also notice I've slipped in a very pretty result. If $S = S_0 + \frac{dS}{d\tau}\tau$, we see that:

$$F = -\frac{1}{3}\frac{dS}{d\tau} .$$

You've seen this before. It's just the diffusion flux, you know this from the heat equation. If $S = \sigma T^4$, then:

$$F = -\frac{1}{3}\frac{\sigma}{\kappa\rho}T^3\frac{dT}{dz} .$$

In general, it means that if the flux is constant for any arbitrary S, then the gradient of the source function increases when κ increases. Another way to say the same thing is that

$$\int_0^\infty \frac{1}{\kappa_\nu}\frac{dS_\nu}{dz}d\nu$$

must be constant for a planar layer, so the relative brightness of an opaque layer must be higher at the surface than for a diffuse, more transparent layer — the edges of very opaque objects look sharper.

Now let's go back to the RTE, this time for a curved layer:

$$\mu \frac{\partial I}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial I}{\partial \mu} = -\kappa I + j \qquad \text{(I'll ignore } \rho \text{ for now).}$$

The radius, not τ, is our independent variable (NB: $\kappa r \neq \int \kappa dr$ in general). We can still define $S = j/\kappa$ and, if κ is independent of μ, or only weakly dependent, the moment-taking process can still proceed as before. Now:

$$F = F_0 r^{-2},$$

as you well know, but for the next moment, we cannot assume $K = \frac{1}{3}J$. As I wrote earlier and said in class,

$$\langle \mu^2 \rangle \equiv \frac{\int_{-1}^{1} \mu^2 I d\mu}{\int_{-1}^{1} I d\mu}$$

but for a curved layer, this is not 1/3 since $I(\tau, 0) \nrightarrow 0$ (the edges of the layer, the tangent points, are not completely dark). We can quantify this using $K - \frac{1}{3}J$, you'll see this for yourselves on taking the next moment of the RTE. This is why we use the luminosity rather than the flux, we can scale $F_0 = (4\pi R^2)^{-1}\mathcal{L}$; the bolometric flux is linked to this quantity:

> Note: Terminology F is the bolometric, F_ν is the monochromatic flux. The corresponding terms are used for the luminosity. Since, in general, we observe only a limited part of the spectrum of any object, a correction from:
>
> $$F_j = \int_0^\infty F_\nu \mathcal{S}_\nu \, d\nu \Big/ \int_0^\infty \mathcal{S}_\nu \, d\nu$$
>
> to F is necessary, called BC or bolometric correction. Here $\mathcal{S}_\nu$ is the response function or "filter sensitivity", how any detector responds to the radiation.

When viewing a body from its exterior, we see light emerging only from its last interaction. Consequently, to understand what we see, think of it as a map in depth scaled by κ_ν. If κ_ν is large, we have τ_ν large even, perhaps, at small τ_R. In a thermally emitting medium, this means we see only the uppermost, cooler layers. If, in contrast, κ_ν is small, we see to deeper layers. Thus, the brightness depends on κ_ν.