# Magnetic Fluids: Alfven Waves etc.

For *magnetized* fluids, we must supplement the equations of motion and continuity with a continuous fluid form for the Lorentz force:

$$\rho \mathbf{a} = \frac{1}{c}\mathbf{J}\times\mathbf{B}$$

assuming a strong coupling between the ions and electrons. That is, in the approximation we'll use here, no net electric fields appear in the medium:

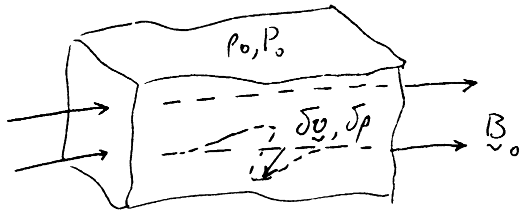
$$\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B} = \mathbf{E}_{int} = 0$$

so any *local* field is generated only by the dynamo-like motion of the fluid itself. Then the Maxwell equations are (sorry but these will be cgs - Gaussian units):

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J}$$

$$\frac{\partial}{\partial t}\mathbf{B} = \nabla\times(\mathbf{v}\times\mathbf{B})$$

$$\nabla\cdot\mathbf{B} = 0$$




and now:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0$$

$$\rho\frac{d}{dt}\mathbf{v} = -\nabla P + \frac{1}{c}\mathbf{J}\times\mathbf{B} + \rho\mathbf{a}'$$

any external field or force, (i.e. gravity)

A useful concept for our future consideration will be the notion of a *very* special sort of field, called "force-free" because $\mathbf{J}\times\mathbf{B} = 0$. Then $\nabla\times\mathbf{B} = \alpha\mathbf{B}$ which, you'll notice, produces for any scalar function $\alpha$ (especially for $\alpha$ = constant) an elliptic equation whose solutions depend only on the boundary conditions and not on time (since $\nabla\cdot\mathbf{B} = 0$ always, we have $\nabla^2\mathbf{B}$ as the spatial operator).

An immediate application of these equations is an analogy with our two earlier acoustic problems: if you have a density fluctuation in a *magnetic* fluid, it should propogate with some characteristic speed in the same way we saw for an ordinary pressure wave. The field, however, has a "tension", the energy density varies differently than for an ideal gas, so depending on how $\mathbf{J}$ couples to $\mathbf{B}$ we may have many more types of waves. Furthermore, and *essentially*, $\mathbf{B}$ is a vector field and can't be isotropic (unless turbulent, and even in *this* case it must be anisotropic at some length scale), so we *must* have a polarized wave. Ok, let's see what happens,

First, as always, *linearize*! Then, $\mathbf{v}\cdot\nabla\mathbf{v}\to 0$ and $\nabla\cdot\mathbf{q}p \to \nabla\cdot(\mathbf{q}_0\delta p + \delta\mathbf{q}\,p_0)$, etc. for any ~~the~~ two quantities $\mathbf{q}$ and $p$ (for instance, $\rho\mathbf{v}$, etc.):

$$\mathbf{J}\times\mathbf{B} \to \mathbf{J}_0\times\delta\mathbf{B} + \delta\mathbf{J}\times\mathbf{B}_0$$

$$\delta P = c_s^2\,\delta\rho$$

and we'll assume finally that ($\rho_0, \mathbf{B}_0, P_0$, etc.) are spatially uniform (at least locally, to show the basic form of the solution) This gives:

$$\nabla\times\delta\mathbf{B} = \frac{4\pi}{c}\delta\mathbf{J} \to -i\,\mathbf{k}\times\delta\mathbf{B} = \frac{4\pi}{c}\mathbf{J}$$

$$\mathbf{k}\cdot\delta\mathbf{B} = 0 \quad \text{(no compressible modes for } \mathbf{B} \text{ alone)}$$

$$\rho_0 i\omega \delta \underset{\sim}{v} = + c_s^2 ik \delta\rho + \frac{1}{c} \delta \underset{\sim}{J} \times \underset{\sim}{B}_0$$

$$i\omega \delta\rho - \rho_0 i \underset{\sim}{k} \cdot \underset{\sim}{v} = 0$$

$$i\omega \delta \underset{\sim}{B} = -i\underset{\sim}{k} \times (\delta \underset{\sim}{v} \times \underset{\sim}{B}_0)$$

so we have a closed system for the perturbations; then

$$\left.\begin{array}{l} \frac{1}{c}\delta \underset{\sim}{J} \times \underset{\sim}{B}_0 \to -\frac{i}{4\pi}(\underset{\sim}{k} \times \delta \underset{\sim}{B}) \times \underset{\sim}{B}_0 \\ c_s^2 \delta\rho \to \rho_0 \frac{c_s^2}{\omega} \underset{\sim}{k} \cdot \delta \underset{\sim}{v} \\ \delta \underset{\sim}{B} = -\frac{1}{\omega} \underset{\sim}{k} \times (\delta \underset{\sim}{v} \times \underset{\sim}{B}_0) . \end{array}\right\} \to i(\underset{\sim}{k} \times \delta \underset{\sim}{B}) \times \underset{\sim}{B}_0 = \frac{i}{4\pi\omega} \underset{\sim}{k} \times \left\{ \left( \underset{\sim}{k} \times \left[ \delta \underset{\sim}{v} \times \underset{\sim}{B}_0 \right] \right) \times \underset{\sim}{B}_0 \right\}$$

Before you get hopelessly lost with all the symbols, notice that we have - from the $\underset{\sim}{J} \times \underset{\sim}{B}$ term, a quantity:

$$\frac{B_0^2}{4\pi\rho_0} \cdot \left( \begin{array}{c} \text{some horrible} \\ \text{vectorial terms} \end{array} \right) ;$$

since $B_0^2$ is an energy density and $\rho_0$ is the mass density, this ratio must have the dimensions of velocity and, in fact, we will give it a name:

$$v_A^2 = \frac{B_0^2}{4\pi\rho_0} \Rightarrow \text{(Alfvèn speed)} .$$

Before I get too detailed, look at the basic result in a way that ignores the polarizations, just term by term:

$$[\omega^2] \ominus [k^2 c_s^2] \oplus [k^2 v_A^2] ;$$

yes - it's a very strange notation but it gets the essentials across! The first $k^2$ term is a longitudinal (compressional) mode, the density variation leading to a simple acoustic wave. The second is much more complicated because of the double vector product relative to $\underset{\sim}{B}_0 = B_0 \hat{b}$, so $\delta \underset{\sim}{v} \times \hat{b}_0 \to \delta v \, \hat{k} \times \hat{b}$ is the angle of propogation relative to the initial, steady $\underset{\sim}{B}_0$. NB: The second term can be either $\geq 0$ or $\leq 0$ !! Thus there are two different speeds for this medium, the so-called fast and slow magnetosonic modes, in addition to $c_s$ and $v_A$. These depend on the angle of propogation so, for instance, a spherical disturbance in a gas with an initial magnetic field will generate a wide spectrum of disturbances (e.g. Kulsrud et al. 1965, ApJ, 142, ~~479~~ 491 as a lovely initial paper, a not-widely read or cited jewel!).

NB: For many astrophysically interesting environments, you'll find $v_A > c_s$. This has several important consequences. We saw that a shock results in a classical gas if $v_\Sigma > c_s$; but now if the medium is strongly coupled (fluid limit) and highly ionized (plasma), a precursor may run ahead of the shock, allowing the gas to adjust its internal conditions before the arrival of $\Sigma$. Thus, instead of producing a discontinuous change, The fluid can slowly adjust in density and pressure – in other words, the medium simply compresses. In a partially ionized medium, the neutral component doesn't feel $\underset{\sim}{B}_0$ at all while the ionized component does, a condition called ambipolar drift, so the neutral gas compresses while the ionized gas doesn't. If these are really different species, not simply two ionization states of the same species, interesting effects occur (but you'll have to look into these yourself, I recommend papers by Hartquist, Caselli, Draine, for instance).

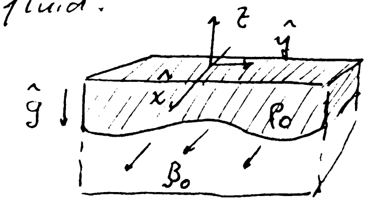
⇒(One last thing here, I'll leave for you the problem of obtaining the angular terms in the wave dispersion relation).

NB As another aside, you can also see how a magnetic field alters both the stability of a gas under gravity and affects shear. Think always of tension; the field guides flows so gradients in $\underset{\sim}{B}_0$, or in perturbations $\delta\underset{\sim}{B}$, cause flows to bend. This, in turn, produces $\nabla\times\delta\underset{\sim}{v}$ and $\underset{\sim}{v}\cdot\nabla\underset{\sim}{v}$ terms, and also introduces (locally) centrifugal forces that act either with or against $\underset{\sim}{g}$, an external gravitational field. Consider a horizontal field $\underset{\sim}{B}_0$ with gravity $g$ such that $\underset{\sim}{g}\times\underset{\sim}{B}_0 = yB_0\,\hat{y}\times\hat{b}_0 \neq 0$. We then have a possible propagation mode that's the magnetic analog of a gravity wave (that is, a sound wave for which the restoring force is both pressure and an external gravitational field – DO NOT mistake this for a gravitational wave, a general relativistic vacuum solution for a time dependent mass distribution). This instability, called the Kruskal – Schwarzschild mode, occurs when a plasma is supported against gravity by a magnetic field and is the analog of the Rayleigh – Taylor instability for a magnetic fluid.

It will take some considerable algebra, but if $\underset{\sim}{B}_0 = B_0\hat{x}$ and we take

$$\underset{\sim}{v} = \begin{pmatrix} 0 \\ v_y(z) \\ v_z(z) \end{pmatrix} e^{i(\omega t - ky)}$$

$$(\partial_t \to i\omega)$$




so $\partial_x \to 0$, $\partial_y \to -ik$, $\partial_z \to \frac{d}{dz}$ as operators, the equations (linearized) for the perturbation (with $v_A^2 = B_0^2/4\pi\rho_0$) become:

$$i\omega\delta\rho + \rho_0 v_z' - ik\rho_0 v_y = 0$$

$$i\omega\delta\underset{\sim}{B} = B_0(ikv_y - v_z')\hat{x}$$

$$\frac{c}{4\pi}(\nabla\times\delta\underset{\sim}{B}) = \delta\underset{\sim}{J} = ik\,\delta B\,\hat{y} - \frac{d\delta B}{dz}\hat{z}$$

$$\frac{\delta\rho}{\rho_0} g\hat{z} = \frac{\omega^2}{\lambda i\omega} v_z' g\hat{z} \qquad \lambda = -\omega^2 + k^2 v_A^2$$

and from the equations of motion:

$$i\omega v_z\hat{z} + \hat{y}\,i\omega v_y = \frac{1}{4\pi\rho_0} ik\,\delta B\cdot B_0\,\hat{y}\times\hat{x} - \frac{B_0}{4\pi\rho_0}\frac{d\delta B}{dz}\hat{z}\times\hat{x} - \frac{\delta\rho}{\rho_0} g\hat{z}$$

so in the limit of $v_A^2 \to 0$ we obtain:

$$-\omega^2 = gk,$$

which is unconditionally unstable ($\omega$ is in conjugate pairs), the main effect here is the difference in the compressibility of the $\underset{\sim}{B}_0$ field and the matter.

In class, I discussed some points about turbulence but it's probably useful to write out a few points; our discussion was very quick. First, a fundamental point:

Turbulence is an intrinsically dissipative process and, as such, cannot be adiabatic (Hamiltonian). This distinguishes it from merely random motions, such as those of point particles in a gravitationally bound system (thanks, Andrea, for asking about this!). It is inherently vortical - that is, $\omega = \nabla \times \underset{\sim}{v}$ dominates the formation of structure - and proceeds from some scale on which the energy is injected down to very small scale.

In the simplest picture, we imagine a fluid to be controlled by two different distribution functions. One is $f(\underset{\sim}{v})$, the microscopic d.f. you've used already. The other, I'll call it $F(\delta \underset{\sim}{v})$, is a macroscopic random variable. Now if:

$$v_i \to V_i + \delta v_i \,, \quad \langle \delta v_i \rangle = 0 \,, \quad \langle v_i \rangle = V_i$$

means averaging over $F$, then

$$T_{ij} \to \rho (V_i V_j + \langle \delta v_i \delta v_j \rangle)$$

for an incompressible fluid (or else we would have $\langle \delta p \, \delta v_k \rangle V_\ell$ terms for some $(k, \ell)$, and even $\langle \delta p \, \delta v_i \, \delta v_j \rangle$ even for the first expansion). For a gaussian $F$, we have $\langle \delta v_i \delta v_j \delta v_k \rangle = 0$ and $\langle \delta v_i \delta v_j \delta v_k \delta v_\ell \rangle$ = permutations of $\langle \delta v_i \delta v_j \rangle \langle \delta v_k \delta v_\ell \rangle$. But if the flow is not gaussian in $F$, then the third moment, $\langle \delta v_i \delta v_j \delta v_k \rangle$ does not necessarily vanish! The founding paper on modern theory is Kolmogorov (1941) (called universally K41) in which he introduced a fundamental notion - a cascade that proceeds from large scale down to one at which $\nu$, the molecular viscosity, dominates. This, according to hypothesis, proceeds in a scale-free manner:

$$\varepsilon = \rho \frac{(\delta v)^3}{\ell} = \text{constant} \to \delta v \sim \ell^{1/3} \varepsilon^{1/3}$$

$\llcorner$ energy dissipation rate ($\dot{E}$ per unit volume)

so that we can introduce a structure function, essentially the energy,

$$\langle (\delta v)^2 \rangle \sim \varepsilon^{2/3} \ell^{2/3}$$

The minimum scale is given by the dimensioned quantities, so since

$$[\varepsilon] = \text{erg}\, s^{-1} \to \ell^{4} t^{-3}$$

$$[\nu] = \ell^2 t^{-1} \qquad \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} = \ell_K = \text{Kolmogorov scale}$$

$$(\nu \varepsilon)^{\frac{1}{4}} = \omega_K = \text{vortical scale},$$

and if $\ell > \ell_K$, the assumption is that the cascade is scale free, proceeding as $f(\ell/\ell_K)$. An alternative way of seeing this is from the concept of marginal stability, that the cascade always proceeds by maintaining $Re \sim (Re)_c$ at all scales so:

$$\frac{v(\ell)\,\ell}{\nu(\ell)} = \frac{v(\ell')\,\ell'}{\nu(\ell')} = \frac{v_K \ell_K}{\nu_0},$$

with $(\varepsilon \ell)^{1/3} = v(\ell)$ we have $\varepsilon^{1/3} \ell^{4/3} (\nu(\ell))^{-1}$ = constant, or $\nu(\ell) \sim \varepsilon^{1/3} \ell^{4/3}$ for $\ell > \ell_K$. This may seem an odd way to view a flow, but we also have

$$\varepsilon \sim \nu \frac{v(\ell)^2}{\ell^2} = \nu_0 \frac{v_K^2}{\ell_K^2}$$

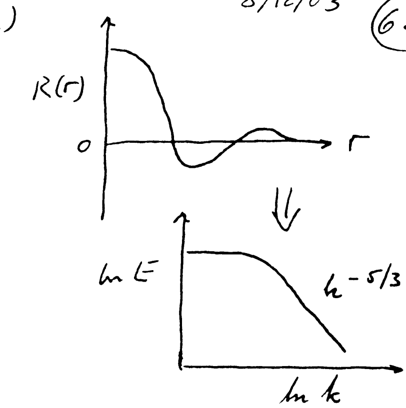
from our calculation earlier using the NSE. The function that describes turbulence is the autocorrelation function

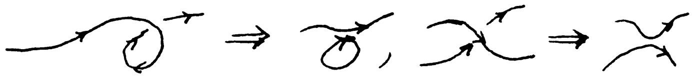
$$R(\underset{\sim}{r}) = \int_{-\infty}^{\infty} \delta v(\underset{\sim}{x} + \underset{\sim}{r}) \, \delta v(\underset{\sim}{x}) \, d\underset{\sim}{x}$$

whose Fourier transform, $\Phi(\underset{\sim}{k})$ is the power spectrum; K41 leads to:

$$E(k) \sim \varepsilon^{2/3} k^{-5/3},$$

The largest power (as you already know from the preceeding discussion) is found on the largest scale, called the <u>source</u> scale (or Taylor scale) for the turbulence. The large scale shear forms the first instability that breaks down through successive growth of the smaller scale structures. You can think of this as a topological process, in the sense that individual vortex structures (filaments, threads) interact locally as a vector field and reconnect:



and other forms are possible.

<u>NB</u> In other words, at this level turbulence resembles magnetic fields in the sense that $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ is analogous to $\mathbf{B} = \nabla \times \mathbf{A}$; note that

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} - \nabla \tfrac{1}{2} v^2 + (\nabla \times \mathbf{v}) \times \mathbf{v}$$

and taking the $\nabla \times$ of the velocity equation for an inviscid flow gives:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{v} \times \boldsymbol{\omega}).$$

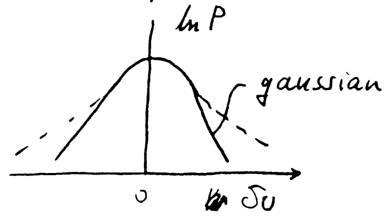
The analogy, introduced in the late '60s by Moffat, is more than formal and <u>very</u> profound – vortex structures can diffuse ($\nu \nabla^2 \omega$) <u>and</u> twist (reconnect) in a manner like magnetic field lines. What does this mean for our discussions of flows and disks? For sheared flows that can become Kelvin–Helmholtz unstable, and this will happen for <u>any</u> disk for which the angular velocity does not increase outward, vorticity will cascade through turbulence to strongly couple the disk material. For our evolution equation, we needed $\bar{\eta}$. The <u>conventional</u> choice is $\bar{\eta} \sim c_s z_0$, but this implies that $z_0$ and $c_s$ are fundamental properties of the cascade, and neither is! But both depend on temperature and, further, if

$$\bar{\eta} \sim z_0 c_s \Sigma \sim \Sigma c_s^2 v_\varphi^{-1} r \sim c_s^2 r^{3/2} \Sigma(r)$$

which is from the thin disk approximation. If $\bar{\eta}$ instead depends on $\nu(\ell_*)$ with $\ell_*$ being the source scale (for example, the epicyclic frequency governing the shear) then we have a <u>fundamentally</u> different coupling between $dT/d\Sigma$ and $\bar{\eta}$ and I'll leave this as an open question.

Today, Fabio asked the question: what is the difference between a gaussian (simple random process) and a turbulent line profile? As you saw in our session with the molecular cloud, called MBM3, the structures seen in turbulence can be highly correlated. But further, a note.

In the laboratory, you normally measure the *temporal* fluctuation spectrum at different positions in one direction, the positions being adjustable. Since the *third* moment doesn't vanish for non-gaussian statistics, the probability of large fluctuation is *greater* than for a gaussian function. In the laboratory, you can express this using a scalar quantity, the probability distribution function (or PDF), formed by averaging all of the fluctuations together and looking at $P(\delta v)$ as a function of $\underset{\sim}{x}$. The broad wings are the signal of turbulence. *But* in cosmic flows, we have only one time for observation ($t_{dynamics} >>$ (human lifetime) in general in *many* environments) so we've actually observed a line profile that averages, at one time, over many turbulent fluctuations in space along a line of sight — in other words, the PDF *is* the line profile! Thus if $\varphi_0(v)$ is the intrinsic line profile and $F(v)$ is the kernel from the turbulence, then:




$$\varphi_{obs}(v) = F * \varphi_0$$

is the *observed* line profile. If $F$ is a power law, $\varphi_{obs}$ will have broader wings than a gaussian (think about the Kolmogorov spectrum) and the difference will be the model for the departures from uncorrelated flows.