Note on vorticity

My apologies for the hurried discussion today. Let me return, in a note, to the point about vorticity. From the equations of motion, you have:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} \to \frac{\partial}{\partial t} + \nabla\tfrac{1}{2}v^2 - \mathbf{v}\times(\nabla\times\mathbf{v})$$

in general. For an ideal fluid, for streamlines, you normally assume $\nabla\times\mathbf{v} = \boldsymbol{\omega} = 0$. But what if it isn't? Then:

$$\nabla\times\left[\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v}\right] = \frac{\partial}{\partial t}\boldsymbol{\omega} - \nabla\times\mathbf{v}\times\boldsymbol{\omega} = -\nabla\times\left(\frac{1}{\rho}\nabla P\right) + \nu\nabla^2\boldsymbol{\omega} + \nabla\times\mathbf{a}$$

From $\nabla\times(f\nabla g) = \nabla f\times\nabla g$, if $P \neq P(\rho)$ (isobaric surfaces corresponding to isodensity – or isothermal surfaces):

$$\nabla\times\left(\frac{1}{\rho}\right)\nabla P = \left(\nabla\frac{1}{\rho}\right)\times(\nabla P),$$

which you'll notice acts like an acceleration for the vorticity. Notice also that a term that resembles the Lorentz force appears, as I'd mentioned in earlier notes. If we assume only slow motions, then we obtain:

$$\frac{\partial}{\partial t}\boldsymbol{\omega} = \nu\nabla^2\boldsymbol{\omega} \quad \text{for \underline{barotropic} cases,}$$

a diffusion equation in which the fluid motion decreases over a time of order ν^{-1}. But if not, slow and if the medium is baroclinic, both $\mathbf{v}\times\boldsymbol{\omega}$ and $\nabla\rho\times\nabla P$ will drive circulation. Now go to a rotating frame:

$$\mathbf{v} \to \mathbf{v} + \boldsymbol{\Omega}\times\mathbf{r} = \mathbf{v} + \Omega\hat{k}\times\mathbf{r} \leftarrow \text{(Coriolis term)}$$

where $\hat{k}$ is perpendicular to the plane of rotation. Then $\mathbf{a} = \mathbf{g} + 2\boldsymbol{\Omega}\times\mathbf{r}$ and $\nabla\times(\boldsymbol{\Omega}\times\mathbf{r})$ is no longer zero, even though $\nabla\times\mathbf{g} = 0$. Thus you have an absolute vorticity even if the motion vanishes and a source to drive instabilities (KH and other vortical modes). Further, any vertical motion on a sphere will drive circulation, so also radial motion in a disk will lead to spirals and vortices! I'll suggest you should think about the Toomre criterion, for instance, in a self-gravitating disk or the Rayleigh–Taylor instability in a rotating system, just for fun.

One more note: Every method we've developed for MHD can be used for $\boldsymbol{\omega}$ and vice versa. The principle difference comes from defining the quantity:

$$\Gamma = \int \mathbf{v}\cdot d\boldsymbol{\ell}, \text{ the circulation} \to = \int \boldsymbol{\omega}\cdot d\boldsymbol{\Sigma} \text{ by Stokes' theorem}$$

and the evolution of another quantity:

$$H = \int \mathbf{v}\cdot\boldsymbol{\omega}\, d\mathbf{x}, \text{ the helicity.}$$

The analogs in magnetism are:

$$\Phi = \int \mathbf{A}\cdot d\boldsymbol{\ell} = \int \mathbf{B}\cdot d\boldsymbol{\Sigma}, \text{ flux} \qquad \mathbf{B} = \nabla\times\mathbf{A}$$
$$H_M = \int \mathbf{A}\cdot\mathbf{B}\, d\mathbf{x},$$

Fields that minimize both the energy and helicity are (for $\mathbf{B}$) the special cases when $\mathbf{J} = \nabla\times\mathbf{B}$ gives $\mathbf{J}\times\mathbf{B} = 0 \to (\nabla\times\mathbf{B})\times\mathbf{B} = 0$ so $\nabla\times\mathbf{B} = \alpha\mathbf{B}$ with α = scalar function. Similarly, for $\mathbf{v}\times\boldsymbol{\omega} = 0$ we have $\nabla\times\mathbf{v} = \alpha_v\mathbf{v}$. The first is a force-free field (the result is Woltjer's (1956) theorem) but the latter is not as simple to understand! This is mainly because $\mathbf{A}$ is a potential but $\mathbf{v}$ is not! The field $\mathbf{A}$ has an evolution equation that depends on $\mathbf{v}$:

$$\partial_t\mathbf{B} = \nabla\times(\mathbf{v}\times\mathbf{B}) + \text{(other terms)}$$
$$\to \partial_t\mathbf{A} = \mathbf{v}\times(\nabla\times\mathbf{B}) + \text{(still other terms)}$$

but $\partial_t\boldsymbol{\omega} \to \nabla\times(\partial_t\mathbf{v})$ just returns us to the equations of motion, and this is almost circular.

Faraday and Plasma Dispersion Effects

With apologies for the brevity of the digression this morning, now I should provide a more complete treatment of the dispersion effects in a magnetized plasma. The equations of the problem assume an incident $\underset{\sim}{E}$ and $\underset{\sim}{B}$ field (both functions of time) and $\underset{\sim}{B}_0 = B_0\hat{b}$. Take

$$\begin{pmatrix} \underset{\sim}{E}(t) \\ \underset{\sim}{B}(t) \end{pmatrix} = \begin{pmatrix} \underset{\sim}{\varepsilon} \\ \underset{\sim}{\beta} \end{pmatrix} e^{i(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{x})}$$

and assume these are perturbations - that the electrons are initially at rest. Then:

$$m_e n_e \dot{\underset{\sim}{v}} = -en_e\underset{\sim}{E} - \frac{e}{c}n_e \underset{\sim}{v}\times\underset{\sim}{B}_0$$

$$-\frac{1}{c}\partial_t\underset{\sim}{B} = \nabla\times\underset{\sim}{E} \rightarrow -\frac{i\omega}{c}\underset{\sim}{\beta} = -i\underset{\sim}{k}\times\underset{\sim}{\varepsilon}$$

$$\frac{1}{c}\partial_t\underset{\sim}{E} = \nabla\times\underset{\sim}{B} - \frac{4\pi}{c}\underset{\sim}{J} \rightarrow \frac{i\omega}{c}\underset{\sim}{\varepsilon} = -i\underset{\sim}{k}\times\underset{\sim}{\beta} + \frac{4\pi}{c}en_e\underset{\sim}{v}e^{-i(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{x})}$$

$$\underset{\sim}{J} = -en_e\underset{\sim}{v};$$

with the linearization, $\dot{\underset{\sim}{v}} = i\omega\underset{\sim}{v}$. Then if $\underset{\sim}{v} = \underset{\sim}{u}e^{i(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{x})}$

$$i\omega m_e n_e\underset{\sim}{u} = -en_e\underset{\sim}{\varepsilon} - \frac{e}{c}n_e\underset{\sim}{u}\times\hat{b}B_0$$

$$\rightarrow \quad i\omega\underset{\sim}{u} = -\underset{\sim}{\varepsilon}\left(\frac{e}{m}\right) - \omega_B\underset{\sim}{u}\times\hat{b}, \quad \omega_B \equiv \frac{eB}{m_e c}$$

$$\frac{i\omega}{c}\underset{\sim}{\varepsilon} = -i\underset{\sim}{k}\times\underset{\sim}{\beta} + \frac{4\pi}{c}en_e\underset{\sim}{u}$$

$$\frac{i\omega}{c}\underset{\sim}{\beta} = i\underset{\sim}{k}\times\underset{\sim}{\varepsilon} \therefore \underset{\sim}{k}\times(\underset{\sim}{k}\times\underset{\sim}{\beta}) = \frac{c}{\omega}\underset{\sim}{k}\times(\underset{\sim}{k}\times\underset{\sim}{\varepsilon})$$

Then:

$$\frac{i\omega}{c}\underset{\sim}{\varepsilon} = -i\underset{\sim}{k}\times(\underset{\sim}{k}\times\underset{\sim}{\varepsilon})\frac{c}{\omega} + \frac{4\pi en_e}{c}\underset{\sim}{u}$$

and:

$$\underset{\sim}{\varepsilon}\frac{i\omega^2}{c^2} = \left[-i\underset{\sim}{k}(\underset{\sim}{k}\cdot\underset{\sim}{\varepsilon}) + ik^2\underset{\sim}{\varepsilon}\right] + \frac{4\pi n_e e}{c}\underset{\sim}{u}\frac{\omega}{c}$$

You'll recognize an old friend, the plasma frequency:

$$\omega_p^2 = \frac{4\pi n_e e^2}{m_e}$$

$$\frac{1}{c^2}\left[\underset{\sim}{\varepsilon}(\omega^2 - k^2c^2) + \underset{\sim}{k}(\underset{\sim}{k}\cdot\underset{\sim}{\varepsilon})\right] = -\frac{4\pi en_e i}{c^2}\underset{\sim}{u}\omega$$

Now take $\underset{\sim}{k}\cdot\underset{\sim}{\varepsilon} = 0$, no longitudinal $\underset{\sim}{E}$ propagation, so that:

$$\underset{\sim}{\varepsilon} = -D^{-2}4\pi n_e e\, i\underset{\sim}{u}\omega, \quad D^{-2} \equiv (\omega^2 - k^2c^2)$$

and thus:

$$i\omega\underset{\sim}{u} = \omega\frac{\omega_p^2}{D^2}i\underset{\sim}{u} - \omega_B\underset{\sim}{u}\times\hat{b},$$

$$i\omega\underset{\sim}{u}\left\{1 - \frac{\omega_p^2}{D^2}\right\} = -\omega_B\underset{\sim}{u}\times\hat{b} \rightarrow \hat{u}\left(1 - \frac{\omega_p^2}{D^2}\right) = i\frac{\omega_B}{\omega}\hat{u}\times\hat{b},$$

which is the equation for a harmonic motion using $\underset{\sim}{u} = u\hat{u}$. Then,

$$\left(1 - \frac{\omega_p^2}{D^2}\right)\hat{u}_1 = -i\frac{\omega_B}{\omega}\hat{u}_2$$

$$i\left\{\left(1 - \frac{\omega_p^2}{D^2}\right)\hat{u}_2\right\} = i\left\{+i\frac{\omega_B}{\omega}u_1\right\}$$

$$\left(1 - \frac{\omega_p^2}{D^2}\right)(\hat{u}_1 + i\hat{u}_2) = i\frac{\omega_B}{\omega}(\hat{u}_1 + i\hat{u}_2) \therefore$$

$$\boxed{\left(1 - \frac{\omega_p^2}{D^2}\right) = i\frac{\omega_B}{\omega}}$$

The plasma dispersion relation since $D^2 = \omega^2 - k^2c^2$.

The electrons have an intrinsic helicity in their motion around $\hat{b}$, the coordinate system I've chosen for the description of their motion: they execute a spiral with some pitch angle with respect to $\hat{b}$, depending on $\underset{\sim}{k} \times \hat{b}$. This means one sense of polarization, right circular, is resonant at $\omega = \omega_B$ while the other, left circular, is not. So the medium displays birefringence - there's a lag of RLP relative to LCP and the plane of an elliptically polarized wave precesses around $\hat{b}$ over propogation distance. Since $k = \frac{n\omega}{c}$, where n is the refractive index, we have:

$$\Delta\varphi \sim \lambda^2 \int n_e \underset{\sim}{B_0} \cdot d\underset{\sim}{l} \ , \quad d\underset{\sim}{l} = \hat{k} dl \rightarrow \lambda^2 \int n_e B_0 \hat{k} \cdot \hat{b}\, dl$$

is the angle of this rotation (you'll find this yourselves, note $\frac{\omega}{2\pi} \lambda = c$).

OK, what does this imply? The angle $\Delta\varphi$ of the rotation is called the Faraday measure and this is the Faraday effect; for scaling:

$$\omega_B \sim 20 \text{ MHz } G^{-1}$$
$$\omega_p \sim 20 \text{ kHz } n_e^{1/2}$$

so in general, $n_e B_0 \sim \omega_p^2 \omega_B$ is a small number relative to ω^3; but at $\lambda \sim$ (a few cm) ($\sim$ GHz) this ratio is compensated by the very long distances encountered in the ISM.

Notice that this rotation depends only on n_e and B_0, not on T. It's therefore strictly independent of temperature (hence, nonthermal) but depends on $\langle \hat{k} \cdot \hat{b} \rangle$, the average angle of propogation. That's why I said in class that if B_0 is turbulent, it's possible to have $\Delta\varphi = 0$ even if $|B_0| \neq 0$.

NB: Any anisotropic medium will produce some polarization, especially from scattering, and the propogation will also be affected by systematics in the anisotropy (for example, dust scattering produces ~~an~~ polarization, but rotation of the plane of polarization results as well from any systematics in the orientation of the grains).