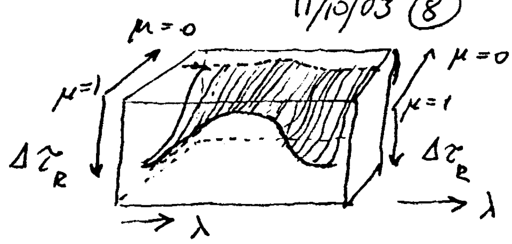

Implication to discussion

So you'll have a record of one of the points I raised in lecture, the one about the map of a spectral line, I didn't explain why it's a surface in the cartoon

VB I'd said you effectively always see down to the same optical depth, $\tau_\lambda \approx 1$. But for the center of a disk, or for vertical observations, this occurs at a greater τ_R than at the limb (edge) or for $\mu \to 0$. In a line, τ_λ / τ_R is larger than at other λ, so you do not see as deep into the layer. The diagram is meant to give that overall impression - the limb generally contributes less to the integrated spectrum, and any quantity that is weighted by the intensity:

$$I(0,\mu) = (1-u+u\mu)\, I(0) \equiv I_0\, \Lambda(\mu)$$

where u is the limb darkening coefficient (we derived this before) that is actually a function of wavelength (i.e. u is really u_λ) so:

$$\langle Q_\lambda \rangle \equiv \int_0^1 Q_\lambda(\mu)\, \Lambda_\lambda(\mu)\, d\mu \Big/ \int_0^1 \Lambda_\lambda(\mu)\, d\mu$$

is the mean of any quantity $Q_\lambda(\mu)$ at the surface. Of course, we should really take $d\Omega$, the solid angle,

$$d\Omega = \mu\, d\mu\, d\varphi$$

but not for the case I'm considering now. In general, however, you need to use (φ, ϑ). Then since τ_R gives the local emissivity, even for scattering, you have limb darkening. If, on the other hand, you deal with a purely scattering layer, then at the limb you may have a bright line, not from T increasing but simply from the pathlength for scattering. We'll come back to this later; I'd mentioned the solar case in class but it may have been too early (the corona problem).

Terminology : The depth of line formation, or continuum formation, is always given by τ_R since you always see into the layer to a maximum τ_λ.

Since in the calculation of $J(\tau)$ we have a reference value for τ_R, hence $J(\tau_R)$, the intensity at any λ is:

$$I_\lambda(0,\mu) = \int_0^\infty J_\lambda(\tau_\lambda)\, e^{-\tau_\lambda/\mu}\, \frac{d\tau_\lambda}{\mu}$$

and

$$\tau_\lambda = \int \frac{d\tau_\lambda}{d\tau_R}\, d\tau_R = \int \left(\frac{\kappa_\lambda}{\kappa_R}\right) d\tau_R$$

for μ fixed; this gives the surface intensity from which we can derive the emitted flux, but for a resolved surface this is the surface brightness. So remember - τ_R is the uniform (λ-independent) depth in the layer and ~~[illegible]~~ we only now know the physical quantities, especially T and P, relative to τ_R.

Terminology : The solution for $T(\tau_R)$ is called the gray atmosphere case since τ_R doesn't depend on λ.

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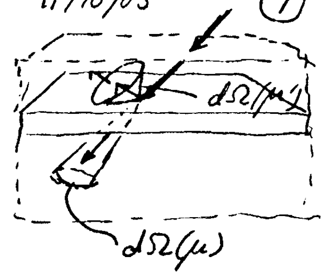
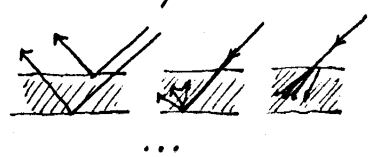
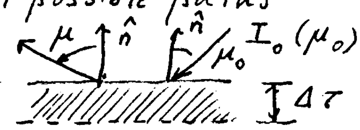


For a final, <u>formal</u> example, let's look at the scattering of a slab of optical depth $\Delta\tau$ in a way that now allows us to add structures more easily – the invariance method. This idea, that it doesn't matter for a thin enough layer how the radiation enters the slab (as shown at right) as long as $\Delta\tau/\tau_{tot} \ll 1$, provides a beautiful way of connecting the continuous solution for scattering (even time dependent) to the



Monte Carlo algorithm and even connects directly to an apparently unrelated problem, the path integral formulation of QED and statistical mechanics. In effect, we follow the path of a photon and then, at the end, sum up all possible paths



⇒ We are, in effect, saying: You have radiation of intensity $I_0(\mu_0)$ coming in and $I(\mu)$ going out; how many different ways can you find for this to happen if <u>only</u> μ matters? Note that I'm not saying this is independent of τ, nor am I saying there's no absorption or emission in the medium. The opacity may include absorption, κ, in which case the re-emission must also be included to insure thermal balance, but this is local (<u>i.e.</u>, it doesn't couple $(\tau-\Delta\tau)$ to $(\tau+\Delta\tau)$, for instance) and for the optically thick case still isotropizes the radiation.

Assume that $\Delta\tau = \sigma\rho\Delta z$. Now we'll neglect $\rho\Delta z = \Delta m$, the mass column density, since in what follows it's just absorbed into σ, ultimately into $\Delta\tau$. Since we will work in the thin layer limit,

$$\exp -\frac{\Delta\tau}{\mu} \xrightarrow[\Delta\tau\to 0]{} \left(1-\frac{\Delta\tau}{\mu}\right), \quad e^{-\Delta\tau/\mu} e^{-\frac{\Delta\tau}{\eta}} \simeq 1-\left(\frac{1}{\mu}+\frac{1}{\eta}\right)\Delta\tau$$

The probability $p(\mu,\mu')$ is defined by $\int_{-1}^{1} d\mu \int_{-1}^{1} d\mu'\, p(\mu,\mu') = 1$ for any arbitrary phase function. Since scattering is conservative, and we assume no reaction on the slab (later this can be modified), what we measure is:

$$I(\eta,\mu) = \left(\frac{1}{\eta}+\frac{1}{\mu}\right)\Delta\tau$$

First, the incident radiation is I_0 that can just reflect so:

$$(1) \quad \frac{I_0(\mu)}{4\eta}\Delta\tau \qquad \left(\underline{\text{NB}} \quad 1-e^{-\Delta\tau/\eta} \simeq \frac{\Delta\tau}{\eta}\right)$$

Then there are two <u>diffuse</u> terms:

$$(2a) \quad \frac{\Delta\tau}{2}\int_0^1 p(\eta,\eta')\frac{d\eta'}{\eta}$$

$$(2b) \quad \frac{\Delta\tau}{2\mu}\int_0^1 p(\eta';\mu)\,d\eta'$$

and finally, there's a second order term having a diffuse scattering of diffusely scattered light:

$$(3) \quad \Delta\tau \int \frac{p(\eta,\eta'')}{\eta''}d\eta'' \int p(\eta';\mu)\,d\eta'$$

The summed terms $(1)\oplus(2a\oplus b)\oplus(3)$ can be taken in any order within $\Delta\tau$, and it doesn't matter if either η or μ is positive or negative.

Writing out the series we require one more assertion:

$$p(\eta, \mu) = p(\mu, \eta),$$

our basic symmetry assumption.

NB: Actually, there are <u>two</u> assumptions at this point, the other is implicit. Notice that none of these various quantities depend on ν. I've already said in class this doesn't hold for Compton scattering, where $\nu(\eta;\mu)$ must be explicitly included, But as a sketch, you can see how this might be included. The intensity $I_\nu(\mu)$ scatters into $I_{\nu'}(\eta)$. Then just as we did (for (2a,b) and (3)) when using diffuse intensity integration, Compton scattering redistributes $\nu \to \nu'$ as a function of $\cos(\varphi - \varphi')$, where φ is an angle in 3D. So we can replace $p(\eta,\mu)$ with $p(\nu';\eta;\nu,\mu)$, treating this as a frequency redistribution function that explicitly depends on angle. The problem is that this is anisotropic and - although the symmetry of the kernel p ~~still~~ still applies, the frequency shift is opposite for forward or backward scattering. (but since $\Delta\nu$ depends on $|\varphi - \varphi'|$ the reversibility of $p(\eta,\mu)$ still holds, in other words).

So combining all terms:

$$p(\eta,\mu)(\eta+\mu)\frac{\Delta\tau}{\eta\mu} = \frac{\Delta\tau_0}{4\eta} + \frac{\Delta\tau_0}{2\eta}\int_0^1 p(\eta';\mu) + \frac{\Delta\tau_0}{2}\int_0^1 p(\eta,\eta')\frac{d\eta'}{\eta'}$$
$$+ \Delta\tau_0\int_0^1 p(\eta,\eta'')\frac{d\eta''}{\eta''}\int_0^1 p(\eta';\mu)\,d\eta'$$

Multiplying by $\mu\eta$ and reducing becomes:

$$p(\eta,\mu)(\eta+\mu) = \frac{\Delta\tau_0}{4}\left(1 + 2\eta\int_0^1 p(\eta,\eta')\,d\eta'\right)\left(1 + 2\mu\int_0^1 p(\eta';\mu)\,d\eta'\right)$$

Now we follow convention, define

$$H(\mu) \equiv 1 + 2\mu\int_0^1 p(\eta,\mu)\,d\eta,$$

a step that was introduced by Chandrasekhar and Ambartsumian, independently, so:

$$H(\mu) = 1 + \frac{\Delta\tau_0}{2}H(\mu)\mu\int_0^1 \frac{H(\eta)\,d\eta}{\eta+\mu}$$

becomes the <u>integral equation</u> solution for the scattering problem.