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An aside about boundary conditions

Assume you have an atmosphere with radiation incident with input intensity I_0 as you just saw for the scattering problem. But now consider what happens at some large distance from the light source. In other words, what happens for a planet?

The central source has a flux, F_ν, that we assume is a function of ν. Now we have $F_\nu(r) = F_\nu(0)\left(\frac{R_0}{r}\right)^2$, for R_0 the radius of the source and r the distance ($F_\nu(0)$ is $F_\nu(R)$). The energy intercepted by a planet of radius R is

$$\dot{\mathcal{E}} = \Delta\Omega \cdot R^2 F_\nu(r)\, \varpi$$

this is the absorbed fraction, the albedo, $A = 1 - \varpi$

which is then re-radiated, ultimately, by the body. This is the condition of thermal equilibrium: whatever goes in, goes out. The input depends on $\sigma T_{\rm eff}^4$ from the luminosity, so:

$$\Delta\Omega \cdot \sigma T^4 = 4\pi\sigma T_{\rm eff}^4\, r^{-2} R_0^2\, \varpi$$

$$\rightarrow \quad T = \varpi^{1/4}\, T_{\rm eff} \left(\frac{R_0}{r}\right)^{1/2} \left(\frac{4\pi}{\Delta\Omega}\right)$$

and we will approximate, for a rapidly rotating body, that $\Delta\Omega \approx 4\pi$ (for a slowly rotating one, use $\pi \le \Delta\Omega \le 4\pi$). Then since $\varpi < 1$, for reasons we'll discuss (but you can guess, there must be some scattering – you see planets and the sky is blue, right?) then $T \ll T_{\rm eff}$.

Now what does this mean? In general, the radiated flux by the planet will occur at $\nu' \ll \nu$ since $T \ll T_{\rm eff}$. If the atmospheric opacity κ_ν at ν is small and ϖ is somehow large (for the Earth [I'll use $\oplus$ for an abbreviation from here on, and $\odot$ for the Sun and related constants] this is because of the ground) then $\kappa_{\nu'}$ may be large relative to κ_ν. For $\oplus$ (and Venus) an important absorber, CO_2, has $\kappa_{\rm visual}$ very small relative to $\kappa_{0.3\mu}$, near the peak of $F_{\nu,\oplus}(T_\oplus)$. So since from our discussion of the diffusion approximation you know that

$$F = \int_0^\infty d\nu\, F_\nu = -\int_0^\infty \frac{1}{\kappa_\nu \rho} \frac{dS_\nu}{dT} \frac{dT}{dz}\, d\nu,$$

assuming $B_\nu(T)$ for S_ν says that when κ_ν is large near the peak of B_ν (or any S_ν), dT/dz and dS_ν/dT must also increase. For $S_\nu \rightarrow B_\nu$, this means that both T and $\frac{dT}{dz}$ must increase relative to the low κ_ν case.

Note: There is no mystery here! Take a trivial example. The nights are now colder. You must sleep under a blanket to stay warm. Your body radiates ≈ 100 W (più o meno) so you lose heat at constant rate (not radiatively, it's conductive). But the blanket reduces the efficiency of heat transport by the air surrounding your body so $T_{\rm air}$ increases at constant internal and external boundary conditions. Thus dT/dz increases and the temperature continues to rise until the heat flux through the blanket reaches equilibrium.

You know this from:

$$\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial z^2} \qquad \text{Fourier eq. in 1D.}$$

Here K is the conductivity so ($T = \mathcal{T}(t)\mathcal{Z}(z)$ and $T(0,z) = \mathcal{Z}_0(z)$ with a constant $T(t, z_1)$ as the outer BC). I'll do this in class, it's enough to know that the solution has a characteristic length scale:

$$\Delta z \sim (Kt)^{1/2}$$

so the temperature after some time t is the same as at $t' < t$ at

$$T(z') = T(z) \rightarrow z = z'\left(\frac{t}{t'}\right)^{1/2}$$

if we assume K to be independent of time. Now this is an integrated quantity, T is a simple scalar function, and is independent of ν as is K. But we can always make the appropriate match between the equations:

$$F = -\frac{1}{3\kappa_R\rho}\frac{dB}{dT}\frac{dT}{dr} = -\underbrace{\left(\frac{4\sigma}{3\kappa_R\rho}T^3\right)}_{K_{rad}}\frac{dT}{dr}$$

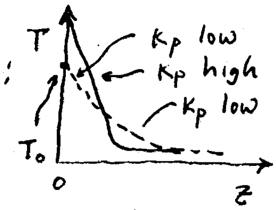
(NB the factor of $1/3$ comes from the Eddington approx.)

This quantity is called the "radiative conductivity" ← K_{rad}

(written in full form you get the equation for the flux in the interior flux:

$$F = -\frac{4ac}{3\kappa_R\rho}T^3\frac{dT}{dr}$$

that applies to, say, the solar — or any stellar — interior). Now back to our discussion.



The effect is vitally important for planetary evolution: this is the so-called greenhouse effect. I'll explain in class in a few weeks why the analogy is not really accurate, but the point is this: since the peak of the radiation is at ν_p, then $j_p = \kappa_p S_p$ is the efficiency for local radiation is large, but since κ_p is large, the loss term is low so T increases until they balance. Since I_0 (or F_0) is constant, if κ_p increases with time so will T, hence the simplest consequence of the changing CO_2 budget — global warming. You see there are two interdependent signatures here: dT/dz and T both increase.

NB: How this works in the $\oplus$-Moon system illustrates my point and returns us to the first class. For $\oplus$, $T_0 \simeq 300$ K (measured at, say, Roma). For the Moon, it is $\lesssim 240$ K. This is very cold relative to room temperature, you know, and yet they're at the same distance. Since in thermal balance, they should have the temperature, the difference (~ 60 K) is entirely due to the $\oplus$'s atmosphere, in particular H_2O, CO, and CO_2. On Venus, $T_0 \sim 710$ K yet, at the cloud top, $T \sim (0.7)^{1/2}\, T_\oplus$ (it has a distance of 0.7 AU from the Sun, by definition $\oplus$ is at 1 AU). The atmosphere is primarily CO_2 with CO, H_2SO_4, SO_2, and no H_2O. In addition, τ_R is enormous even if $\tau_{vis} \ll \tau_R$ (but $\tau_{vis} \gg 1$) so for the lower Venus atmosphere we can use the diffusion approximation and $K = \frac{1}{3}J$. The Venus case, a "dry greenhouse", is also important for $\oplus$ because the clouds have relatively low ϖ_{vis}, hence most of the incident radiation is scattered, that's why the planet is so bright.

Notice that the radiative peak ν is higher as T increases, so eventually the as peak shifts, the opacity drops and the radiative losses stabilize.

Questions: what happens to $T_0(\oplus)$ if the T_{eff} for the $\odot$ increases? What are the effects of clouds? What happens if you move $\oplus$ closer to the $\odot$? (OK, it's your chance to write some science fiction or theology ?!).