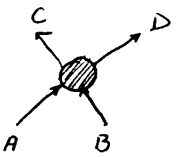
20/10/03

ỳ one point from today. For any process:

$$A + B \overset{LTE}{\underset{Q}{\rightleftharpoons}} C + D$$



$$\frac{N_C N_D}{N_A N_B} = \frac{Z_C Z_D}{Z_A Z_B} e^{-Q/kT}$$

(ex: $^3He + {}^2D \rightleftharpoons {}^4He + p$) although the reverse reaction is very hard to achieve

where, for now, Q is an excitation energy. But each interacting ~~com~~ constituent may have internal degrees of freedom (modes other than translation):

$$Z_j = Z_j^{int}(T)\, Z_j^{trans}(T)$$
$$= Z_j^{int} \cdot (2\pi m_j kT)^{3/2} h^{-3} .$$

so we will always have: (NB I mis-spoke in class, the $T^{3/2}h^{-3}$ factor is absent since all trans. are $T^{3/2}h^{-3}$)

$$\frac{N_C N_D}{N_A N_B} = \frac{Z_C^{int} Z_D^{int}}{Z_A^{int} Z_B^{int}} \left(\frac{m_C m_D}{m_A m_B}\right)^{3/2} e^{-Q/kT}$$

in equilibrium. For molecules (e.g. $C + O \rightleftharpoons CO$), for example:

$$\frac{N_C N_O}{N_{CO}} = \frac{Z_C(T) Z_O(T)}{Z_{CO}^{rot} Z_{CO}^{vib} Z_{CO}^{el}} \left(\frac{m_C m_O}{m_{CO}}\right)^{3/2} \frac{I^{3/2}}{h^3} e^{-D_{CO}/kT} ,$$

I'm using D_{CO} as the dissociation energy of CO and assuming, for instance, that:

$$Z_C = \sum_j^1 g_j^{(c)} e^{-\varepsilon_i^{(c)}/kT}$$

but using $(2J+1)$ as the statistical weight of the J^{th} state and $(v+\frac{1}{2})$ for the vibrational states, we still must compute the exact $E(J,v)$ including all anharmonic effects. If we use $J(J+1)$ for the energy and v for the vibration energy, then, e.g.:

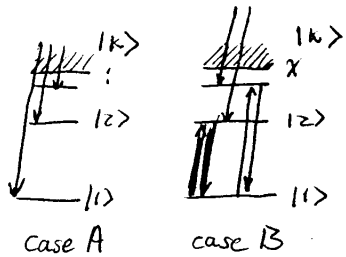
$$Z_j^{rot} \propto \sum (2J+1) e^{-J(J+1)\varepsilon_0/kT} \qquad \varepsilon_0 \to \frac{\hbar^2}{2I}$$

(note: only in the case of a diatomic molecule is this so simple; for complex molecules we must include all the couplings, and possibly three moments of inertia, so Z_j^{int} may not separate - vibrotational bands - as I've done here. Also, don't forget there are electronic states to consider, Z_j^{el}).

If we do not have LTE, this simple approach must fail! In the most general case, called non-LTE or NLTE (a term I will use now repeatedly), we must solve all the rate equations simultaneously - although still for $n_j^* = 0$:

$$-\beta_{ik} n_i^* + \alpha_{ki} n_e - \sum_{j \neq i} C_{ij} n_e n_i^* + \sum_j n_j^* C_{ji} n_e + \sum_{j>i} n_i^* B_{ij} I_{\nu_{ij}} + \sum_{j>i} n_j^* (A_{ji} + B_{ji} I_{\nu_{ij}}) = 0 .$$

Let me return now to a point about α. The recombination rate for an atom, α_{tot}, is summed over all possible ways of having the electron get attached to the ion. If the ground state is accessible directly, then $K \to |1\rangle$. However, at each step down a photon is emitted. It's possible that the level population is so large for some $|j'\rangle$ above the ground state, or for $|1\rangle$, the ground state, that this transition is optically thick.



First, to recapitulate our discussions from today's class: 27/10/03

For a classical free electron, the induced motion is governed by $m_e \ddot{\underset{\sim}{x}} = e\underset{\sim}{E}$, implying that for any frequency ω, the response is $-m_e\omega^2 e\underset{\sim}{x}_\omega = -m_e\omega^2 \underset{\sim}{d}_\omega = e^2 \underset{\sim}{E}_\omega$ for the Fourier transform. Since the equation of motion is linear, each mode is separate and the total motion follows from linear superposition. But you already know that:

$$\frac{d\mathcal{E}}{dt} \equiv P \sim |\ddot{\underset{\sim}{d}}|^2$$

is the power radiated by a moving charge – the Larmor formula – so it follows immediately that σ, since:

$$P = \sigma I$$

is the definition of the power, and $I \sim |E_\omega|^2$, thus in this case

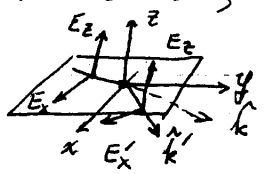
$$\sigma \equiv \sigma_T = \text{cst.}$$

(I'll leave you the exercise of computing the constant). This isn't a surprise, you know, because for a free charge there's no characteristic frequency, no ω_0. For a low enough ω, $\hbar\omega \ll m_e c^2$ so you expect no lag between the ~~force~~ driver ($\underset{\sim}{E}$) and the response ($\underset{\sim}{d}$). Of course, as we've discussed, this is not the case for the Compton effect, but we'll get to that.

For estimates, note that $\sigma_T \sim 6\times10^{-25}\,\text{cm}^2$ or $\sim 0.4\,\text{cm}^2\,\text{g}^{-1}$. For many conditions, $\tau_T = \sigma_T n_e l$ is large enough to affect radiative transfer, in particular the case of a stellar interior (for the ⊙, for example, $\tau_T \sim 1$ so even for no metals, so the minimum opacity, electron scattering is an important opacity source). The Rosseland opacity, κ_R, is independent of both T and ν in this case and, although the radiative diffusion coefficient is dependent on T, it is very simple:

$$D_{rad} \sim \frac{T^3}{\kappa_R} \to \frac{T^3}{\sigma_T}.$$

The radiation isotropizes if τ_T is large but there is no heating of the gas by the radiation, we must get the temperature gradient only from the fact that the pressure is large and the escape probability is reduced. Furthermore, now we have the possibility that the radiation is polarized since, if we look at the last scattering surface – the photosphere – the smallest departure from a spherical symmetry will show up (since $|\underset{\sim}{E}|^2$ is unaltered).



NB This effect was predicted for hot stars (O and B type) in the 1940's by Chandra as a consequence of rapid rotation. It was the inspiration for early work on polarization and led to the accidental discovery of dust scattering in the interstellar medium by Hiltner and Hall.

It's fundamental to note, however, that scattering will produce a dynamical coupling with the radiation even though there's no heating. The effect is easy enough to see this way. Each scattering produces a kick, Δp, in the momentum p. Take $\Delta p \sim p$, which is $h\nu/c$ for a photon. The number of photons incident per second on an area σ_T is:

$$\dot{n}_\gamma = \sigma_T \frac{F_\nu}{h\nu}$$

so the radiative ~~pressure~~ acceleration is:

$$g_{rad} \sim \int \sigma_T F_\nu \frac{d\nu}{h\nu} \cdot \frac{h\nu}{c} \sim \frac{1}{c}\int \sigma_T F_\nu\, d\nu \sim \frac{\sigma_T}{c} F.$$

This is a very interesting result – really! – because it can produce a levitation: if we look at the single scattering case, then each electron feels g, a gravitational force, if it's bound. But $g = \frac{GM}{r^2}$

and $g_{rad} \sim \frac{L}{4\pi r^2}\frac{\sigma_T}{c}$ so if:

$$g = g_{rad}(L_*)$$

we have a critical dynamical condition:

$$L_* = \frac{4\pi GMc}{\sigma_T} \equiv L_{Edd} \sim 10^5 L_\odot \text{ for } \sigma_T$$

independent of the distance! This result, called the Eddington luminosity, was first derived for a different problem of stability but the meaning is the same. For any mass M, if L is large enough, the medium will be unstable to outflow. Why outward motion? Think of what L is – the photons, even if only diffusively, move outward (again, recall that's what a surface does for a radiating body). Note, however, that using 4π as the solid angle requires complete covering of the source, and this is computed for σ_T. In general:

$$g_{rad} \sim \frac{1}{c}\int_0^\infty \kappa_\nu F_\nu \, d\nu$$

which, if we define a new mean opacity:

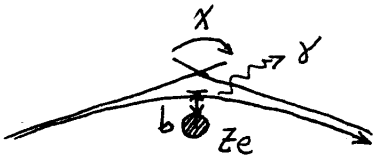
$$\kappa_P = \frac{1}{F}\int_0^\infty \kappa_\nu F_\nu \, d\nu \equiv \text{Planck mean}$$

gives us a way to evaluate the dynamical stability of the gas. For $\sigma_\nu = \sigma_T = \kappa_\nu$, $\kappa_P = \kappa_R$, but this is not generally true.

Note also that if κ_ν is dependent on ν, and of course so is F_ν, then the maximum of the flux will weight most heavily the Planck mean. But, looked at the other way around, the stability of the medium is undermined by a gas for which the largest opacity most closely coincides with $\max(F_\nu)$. This is why L_{Edd} is a limiting luminosity, σ_T is gray (no ν-dependence). Anything that increases κ (i.e. κ_R) decreases L_*, in effect further destabilizing the medium. The radiation pressure in an opaque medium is also related to the energy density by:

$$P_{rad} = \tfrac{1}{3}\rho c^2 \quad \text{(remember K/J?)}$$

(so you see that this also relates to F_ν, since $\int \frac{1}{c} F_\nu \, d\nu$ is clearly related dimensionally – again I'll leave it to you to figure out the details), it's enough to say that τ_ν, although dimensionless, is a measure of the efficiency of this coupling so as the number of scatterings increases, so does P_{rad}).



We now go to the second continuum process, this one actually deriving from the ionization of the medium. We take an electron moving past an ion of charge Ze. Then the distance of closest approach, b (called, as I'd said in class, the impact parameter) is:

$$\tfrac{1}{2}m_e v^2 = \frac{Ze^2}{b} \rightarrow \quad \pi b^2 \equiv \sigma = \left(\frac{2Ze^2}{m_e v^2}\right)^2 \pi$$

The mean free path, λ, gives a collision rate $t_c^{-1}(v) = n_e \sigma(v) v$ for each impacting electron so if $f(v)$ is the velocity distribution function for the electrons,

$$t_c^{-1} = n_e \langle \sigma v \rangle \equiv n_e \frac{\int \sigma(v) v f(v) \, dv}{\int f(v) \, dv}$$

and the rate of energy release (radiation) per impact is:

$$\dot{\varepsilon} \sim \varepsilon t_c^{-1}(v).$$

Now another estimate comes from taking the momentum change

$$\Delta p = \int_{-\infty}^{\infty} \frac{Ze^2\, dt}{b^2\left(1+\left[\frac{vt}{b}\right]^2\right)} = \frac{b}{v}\cdot\frac{Ze^2}{b^2}\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \frac{Ze^2}{bv}\pi$$

for each impact, so $n_e v \cdot 2\pi b\, db$ is the rate of incidence in a cross section and we then include the crossing time v/b. The change in the energy is $(\Delta p)^2$ so now taking the average over $f(v)$ gives

$$j \sim T^{1/2} \ln\left(\frac{b_{max}}{b_{min}}\right)$$

where $b_{(min,max)}$ are the minimum and maximum impact parameters, also called the Coulomb integral.