So now let's go a bit deeper, after you've recovered from today's "didattica". The point of the long digression was, as always, to emphasize the importance of scaling arguments in most astrophysical arguments - this will be clearer when we get to hydrodynamics.

The production of radiation by a charged medium, bremsstrahlung, is one of the most general radiative processes in astrophysics. Any ionized, optically thin plasma will emit in this manner, more or less, and it competes with all other thermal processes. In this sense, whether in stellar envelopes or galaxy clusters, such emission is the vital plasma diagnostic.

The process is simply:

$$\mathbf{p} \to \mathbf{p}' + \frac{\hbar\omega}{c}\hat{k}$$
$$\varepsilon \to \varepsilon' + \hbar\omega$$

and occurs as a random walk of an electron among the ions of a plasma. As such, $\Delta p = \Delta p_\perp$ is:

$$\Delta p_\perp = \int_{-\infty}^{\infty} F(t)\,dt = \int_{-\infty}^{\infty} \frac{Ze^2\,dt}{r^2(t)}$$

$$r^2(t) = b^2 + v^2t^2, \quad b = \text{impact parameter.}$$

Then we have:

$$\Delta p_\perp = \frac{Ze^2}{b^2}\frac{b}{v}\int_{-\infty}^{\infty}\frac{d\xi}{1+\xi^2} = \frac{\pi Ze^2}{bv}$$

and:

$$\langle(\Delta p_\perp)^2\rangle = \left(\frac{\pi Ze^2}{b}\right)^2\left\langle\frac{1}{v^2}\right\rangle$$

where we'll take the average over the thermal distribution. But before we do this, we need to consider the rate of collisions; this depends on the rate of arrival of electrons:

$$2\pi b\,db\,n_e v$$

so we take the rate of emission to be:

$$j \sim \iint (\Delta p_\perp)^2 f(v)\, n_e^2\, b\, db\, dv \sim Z^2 n_e^2 \ln\left(\frac{b_{max}}{b_{min}}\right)\int_{v_{min}}^{\infty} e^{-\frac{mv^2}{2kT}}\, v\, dv \Big/ \int_0^{\infty} f(v)\, dv$$

the v_{min} coming from the energy condition: to create a photon with energy $\hbar\omega$ we must have at least some kinetic energy - there's a cutoff at $\frac{1}{2}m_e v_{min}^2 = \hbar\omega$. Thus we're really just taking the average value of $\langle v^{-1}\rangle$ with a lower cutoff:

$$j_\omega \sim Z^2 n_e^2 \left\langle\frac{1}{v}\right\rangle \ln\left(\frac{b_{max}}{b_{min}}\right) e^{-\hbar\omega/kT}$$
$$\sim Z^2 n_e^2\, T^{-1/2}\, \Lambda\, e^{-h\nu/kT}$$

renaming:

$$\Lambda \equiv \ln\frac{b_{max}}{b_{min}}$$

and asserting that this is a weak function of velocity (it will vary slowly since it's only the logarithm). Then taking the integral over frequency,

$$j = j_0\, Z^2 n_e^2\, T^{1/2}.$$

This is the expected emission from a thermal plasma. Now since

$$\frac{j_\nu}{\kappa_\nu} = B_\nu \sim \nu^3/(e^{h\nu/kT}-1)$$

we have:

$$\kappa_\nu \sim Z^2 n_e^2 \nu^{-3}(1-e^{-h\nu/kT}),$$

noting that the second term you've seen before when we wrote down the correction for stimulated emission from the transition probabilities:

$$\kappa_\nu \sim B_{kk'} n_k - B_{k'k} n_{k'} \sim B_{kk'} n_k (1 - e^{-h\nu/kT}).$$

Recall we already have a connection between $A_{k'k}$ and $B_{k'k}$ (here k' and k are states in the continuum) such that:

$$A_{k'k}/B_{k'k} \sim \nu^3 \qquad \text{(recall the relation between } A_{ji} \text{ and } j_\nu)$$

for any thermal process so we have:

$$\kappa_\nu \sim Z^2 n_e^2 \nu^{-3} T^{-1/2} (1 - e^{-h\nu/kT}) \Lambda.$$

Thus, for $\nu \ll \frac{kT}{h}$, we have (also recall that the statistical weight of the continuum depends on the momentum, $p^2 dp/h^3$):

$$\kappa_\nu \sim Z^2 n_e^2 T^{-3/2} \nu^{-2}$$

and noting that we have the same cutoff as before, the integrated value of opacity is:

$$\kappa = \kappa_0 n_e^2 T^{-7/2},$$

the Kramers opacity. Actually, this is also found as I'd mentioned in class from noting that:

$$\frac{j}{\kappa} \sim T^4 \quad \text{so} \quad \kappa \sim n_e^2 T^{-7/2}.$$

Thus the emission increases and the opacity decreases with increasing temperature — and the optical depth also depends on frequency. Unlike electron scattering, here the induced dipole is due to a scalar field and the emission depends on the kinetic energy of the particle, and its impact parameter (in other words, the high frequencies are emitted from collisions with large $\Delta p_\perp$, so small b; those of low frequency come from large b). For an optically thin gas, <u>at some frequency</u>, there <u>must</u> be a lower cutoff at which the emissivity follows a Rayleigh-Jeans law, $B_\nu \sim \nu^2$, while at high frequency there is an exponential cutoff because we've simply not sampled the total electron distribution - some don't radiate and this fraction increases as ν increases for fixed T since $\langle \varepsilon \rangle = kT$.

(NB An alternate derivation comes directly from the Larmor formula, noting that a simpler form for the acceleration is to note that

$$|\ddot{\underset{\sim}{d}}| \sim |\tfrac{v}{b} \Delta v_\perp|$$

for the dipole moment.)

<u>Some additional points</u>: collective motions etc.

We can also ask whether collective modes are possible, in which the plasma emits or absorbs at a characteristic frequency. We've used the crossing time for a lower cutoff, and invoked λ_{Debye} as the screening distance. Now imagine an electron in a cold plasma. This will not radiate but it can absorb. Then:

$$m_e \ddot{\underset{\sim}{x}} = -e\,\delta \underset{\sim}{E} \qquad (\delta \underset{\sim}{E} \text{ is the perturbation creating a weak local field for a displacement } \underset{\sim}{x})$$

so if we take $\delta E = \frac{\partial E}{\partial x} x$ for one dimensional motion,

$$\frac{\partial E}{\partial x} = 4\pi e n_e \rightarrow \ddot{x} = \frac{4\pi e^2 n_e}{m_e} x$$

is a harmonic oscillator with $\omega_p^2 = \frac{4\pi n_e}{m_e} e^2$, the <u>plasma frequency</u>.

We'll need this later, for now its enough to say that even a cold plasma will have a response at this frequency, it's a lower cutoff for even the bremsstrahlung and you'll notice it's independent of temperature. It is also then clear (I hope) that even for electron scattering collective effects can be induced; above ω_p the plasma won't respond, below ω_p it will redistribute the energy into the kinetic modes of the gas.