We now treat, as I began in class, the problem of bremsstrahlung. Using the definition of the change in the momentum for a collision:

$$\Delta p = \int_{-\infty}^{\infty} \frac{dt}{\left(1+\frac{v^2t^2}{b^2}\right)} \cdot \frac{Ze^2}{b^2} = \frac{Ze^2}{bv}\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$$
$$= \frac{\pi Z e^2}{bv}$$

so the nr. of collisions $dN = 2\pi b n_e v\, db \rightarrow 2\pi\left(\frac{\pi Z e^2}{bv}\right)^2 N_Z n_e\, v\int_{b_{min}}^{b_{max}}\frac{db}{b}$. (This is where I started the long digression on the process of approximating astrophysical processes. But that was likely confusing and I wanted to be sure you had something more detailed in writing).

So the energy of a particle, ε, is changed by $\Delta\varepsilon = \frac{1}{2m}(\Delta p)^2$ which then must be integrated over the velocity distribution. First, we have:

$$\Lambda \equiv \ln(b_{max}/b_{min})$$

which, called the Coulomb integral, is a quantity that describes the screening in the plasma. At b_{max}, the charge Ze is effectively screened by the background charge while at b_{min}, we have the same condition I first used to estimate the distance of closest approach from the potential. Notice, by the way, what we discussed, that $\Delta p = \Delta \underset{\sim}{p}\cdot\hat{b} = \Delta p_\perp$ and that $\langle(\Delta p)^2\rangle$ doesn't vanish – this collisional development of a trajectory varies like a random walk since while $\langle \Delta p_\perp\rangle$ vanishes for a non-magnetized (isotropic) plasma, $\langle(\Delta p_\perp)^2\rangle$ doesn't. So now taking this average and assuming Λ varies slowly with v, we have:

$$j \sim Z^2 e^4 n_e^2 T^{1/2}$$

(you can work out the constant) for a pure hydrogen plasma (if not, n_e^2 is replaced by $n_i n_e$). Since j is nearly independent of frequency:

$$B_\nu(T) = \frac{j_\nu}{k_\nu} \rightarrow B(T) = \frac{j}{k}$$

and since $B(T) = aT^4$ and $j = j_0\rho^2 T^{1/2}$ (changing now to mass density), then:

$$\kappa = \kappa_0\rho^2 T^{-7/2},$$

one of the earliest results for the opacity of ordinary, ionized matter – the Kramers opacity law. Notice that this means the opacity decreases with increasing temperature if the principal source is bremsstrahlung; we'll soon see the consequences of this for the stability of the matter.

<u>Aside</u> : spectral distributions and energy distributions

As I'd mentioned in class, if a single particle radiates with some power (efficiency), $P(\varepsilon)$, at an energy ε, then

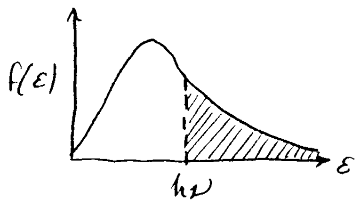
$$P_{tot} = \int_0^\infty f(\varepsilon)\, P(\varepsilon)\, d\varepsilon$$

where $N(\varepsilon)$ or $f(\varepsilon)$ is taken to be the energy distribution. If $f(\varepsilon)$ is thermal, then

$$\int_0^\infty f(\varepsilon)\, d\varepsilon = (2\pi m_e kT)^{3/2},$$

but more to the point you know that $f(\varepsilon) \to 0$ as both $\varepsilon \to 0$ <u>and</u> $\varepsilon \to \infty$. On this assumption, there <u>must</u> be a maximum somewhere in $\varepsilon \in [0, \infty]$ and this <u>must</u> be a function <u>only</u> of T; the dispersion is also only dependent on T. In addition, since the integral is finite, this is normalized over the whole range. We used this for the Boltzmann-Saha equation, $Z^{trans} \sim \frac{T^{3/2}}{h^3}$.

Now we also know, when computing the rate of incidence, that $\varepsilon \geq h\nu$ for any photon of frequency ν; if we see radiation at a frequency ν, then only the portion of the distribution with $\varepsilon \geq h\nu$ can contribute and this is always finite; for $\nu \to \infty$, $F(\varepsilon \geq h\nu) \to 0$ (here:

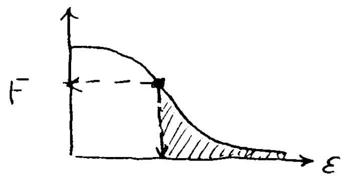


$$F(\varepsilon \geq h\nu) = \int_{h\nu}^\infty f(\varepsilon)\, d\varepsilon < \infty$$

is the cumulant distribution, for a maxwellian:

$$F(x \geq x_0) = erf_m(x_0) \equiv \int_{x_0}^\infty e^{-x^2} x^4\, dx$$

$$erf(x_0) \equiv \int_{x_0}^\infty e^{-x^2}\, dx,$$



which is just the probability of finding any particle above some energy, and this saturates for $h\nu \to 0$. Thus, we expect a relatively flat energy distribution at low ν and an exponential cutoff for high ν.

This is <u>not</u> true for the case of a <u>non-thermal</u> distribution. For nonthermal laws, e.g. a power law:

$$\lim_{\substack{x_0 \to 0 \\ x_1 \to \infty}} \int_{x_0}^{x_1} x^{-s}\, dx \to \infty$$

which is clearly nonsense! Instead, we <u>must</u> specify the lower cutoff for $s > 1$ and the upper cutoff for $s \leq 1$, both of which must be supplied from elsewhere. Thus, the assumption of a power law produces a power law spectrum, since in general (from the Larmor formula) the radiated power for a single particle varies as

$$P(\varepsilon) \sim \varepsilon^n \qquad (\underline{n} \text{ depends on the process})$$

and we <u>still</u> require $\varepsilon \geq h\nu$, whatever the process! Only for <u>bound</u> spectra will be get cutoffs, in general we have:

$$P_\nu \sim \nu^{n-s+1} \sim \nu^{-\alpha} \qquad (\alpha > 0)$$

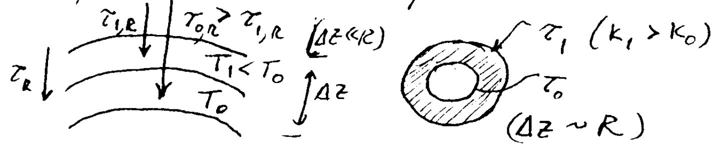
for such processes. Being nonthermal, α <u>doesn't</u> depend on T.

$\Rightarrow$ Remember, what you're sampling at any observed ν is that part of the spectrum at higher energy. Hence, any time you move to lower ν, you're sampling – on average – an older population of radiating particles, those that move from $\varepsilon \to \varepsilon - h\nu$.

To return to the point connecting the optical depth and frequency, we find for any thermal medium that $\tau_\nu \sim \nu^{-2} n_e^2 l$. This is very interesting from the connection it provides with our discussion of atmospheres. Repeatedly, both in class and in these notes, I've said you see into about $\tau_\nu = 1$ at all frequencies. For a thin atmosphere (I mean geometrically thin) the implication is that if $\kappa_\nu \sim \nu^{-n}$, at lower frequencies you see the surface at lower $N = \int n_e \, dl$, the column density (well, for now I'll use this).

Since in terms of κ_R, this is at $d\tau_R = \frac{\kappa_R}{\kappa_\nu} d\tau_\nu \sim \frac{\kappa_R}{\kappa_\nu}$, if $\kappa_\nu \uparrow$, $\tau_R \downarrow$ so you're sampling light emerging from smaller τ_R. This layer cools more easily than the lower layers so, in general, $T(\tau_\nu) \downarrow$.

This is expressed, for a geometrically thin layer by saying that at smaller κ_ν the brightness of the source is higher.



If the layer is both uniform density and optically thick, <u>and</u> if somehow we also have the same T, then the intensity is independent of wavelength (or nearly so). On the other hand, if we have a very large region, and between two frequencies ν_0 and ν, there's a large difference in the geometric depth to which we observe, then at high frequency we see a <u>smaller</u> surface area (smaller radius) – the <u>size</u> of the source changes with frequency! This is a very general result and has an important effect: it changes the flux distribution of the emergent light.

How this happens can be seen simply. You'll recall from our definition for the flux from the spherical RTE that:

$$F = \frac{L}{4\pi r^2} \rightarrow \frac{F_0 \cdot A_0}{4\pi r^2}. \quad (A_0 = \text{area of source})$$

When we did this, we assumed $A_0 = 4\pi R_0^2$ with the radius $R_0 = $ constant. <u>But</u> if κ_ν depends strongly on ν, then $R_0 \rightarrow R_{0,\nu}$ so

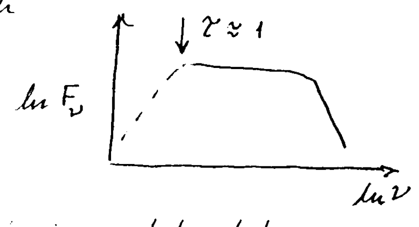
$$F_\nu = F_{0,\nu} R_{0,\nu}^2 / r^2.$$

Thus even if $F_{0,\nu}$ is independent of ν, F_ν will change with frequency because of a change in the area of the surface from which the light emerges, even if the density and temperature are both constant! If the density ~~also~~ also changes, the effect is exaggerated since ~~[illegible]~~ $\kappa_\nu \sim n_e^\alpha \nu^{-n}$ (α, in general, is either 1 or 2) and for many cases, $n_e \sim r^{-2}$ or something like this (we'll see how in a short time, I promise). The precise form of the law that results depends on how the radiation emerges, so recall that:

$$I_\nu(0) = \int S_\nu(\tau) e^{-\tau_\nu} d\tau_\nu$$

and you see that $\kappa_\nu S_\nu = j_\nu$ by definition and $d\tau \sim \nu^{-n} n_e^\alpha dr$ then ~~[illegible]~~ for $n_e \sim r^{-s}$, $\int n_e^\alpha \nu^{-2} dr \sim \nu^{-2} r^{-(\alpha s)+1}$. But $n_e^{+1/s} \sim r^{-1}$ so you can rewrite this integral in terms of the density. I won't continue this now – you need a clearer idea of how to obtain $\underline{s}$ in $n_e \sim r^{-s}$, and that requires hydrodynamics – but you see where we're going.

As I'd said in the lecture, the flux changes <u>even without</u> this effect: at <u>some</u> frequency, the source becomes optically thick and this frequency depends on the column density, N. So you have a way of estimating the density by a measurement, for even constant n_e, of the "turnover frequency" at which the spectrum varies as ν^2, the Planck function.



For nebulae, e.g. H II regions, this is used to determine parameters from <u>radio</u> observations since, in general, $n_e \sim 10^4 - 10^6 \text{ cm}^{-3}$,