23/2/04 ①

Now we can begin the notes for this course! So to recapitulate the discussion from today's lecture, I'll note the steps in the development of the dynamical picture of the Galaxy:

1. global structure – a flattened stellar distribution but not symmetric around the sky. Herschel – star gaging (magnitude distribution) leads to a very non-
1785 symmetric distribution, dominated by relatively local obscuration (after Galileo first resolves the Via Latte and both Orion and the Pleiades).
2. determination of direction (apex) of solar motion, again by Herschel, along with the discovery of angular motion of resolved binary stars and magnitude
1783 dispersion among components (but without calibration); reflection of the solar motion and an indication of the need for a local standard of rest (LSR).
3. discovery (Messier, Herschel) of clustering, especially globular clusters (very compact systems) that are more nearly isotropic than the bulk of the
< 1820 stellar population.
4. determination of stellar parallax and accumulated data on angular (proper)
~1850 motions (Bessel, F. Struve, etc.); introduction (esp. Secchi and Vogel, later Pickering and Maury) of spectral classifications (by 1910).
5. first studies of radial velocities, determination of space motions, introduction of statistical methods for study of stellar motions (Charlier, Schwarzschild, Eddington) (< 1920).

This last step step saw the introduction of a triaxial Gaussian distribution for the relative motions:

$$f(U,V,W) \sim \prod_{i=1}^{3} \exp - \frac{(v_i - \langle v_i \rangle)^2}{2\sigma_i^2} \quad , \quad \sigma_u \neq \sigma_v \neq \sigma_w$$

with two implicit assumptions – each direction is "relaxed" in the sense that $\langle v_i v_j \rangle = \langle v_i^2 \rangle \delta_{ij}$ and all moments higher than the second moment vanish.

6. Determination of distance to Galactic center – not at the Sun – by Shapley using geometry of the globular cluster distribution (< 1920).

This implicitly assumes these clusters are tracing the same gravitational potential as the stars despite their position beyond the location of the ☉. We'll see later the importance of this step for dark matter distributions.

7. Large scale systematic motions and vertex deviation, star streaming (Kapteyn < 1922).

If the ☉ is not central and around this location there are systematic motions, a flattened system must be axisymmetric (to first approximation) and thus a "super-Keplerian" motion may apply.

8. Interstellar extinction (Wolf, Malmquist, Barnard) based on star counts (extension of Herschel's methods) and reddening (differential, spectral) extinction (Trumpler < 1935).

This finally reveals the role of an intervening medium, first identified in the gaseous (absorption line) component by Vogel and Hartmann, in possibly affecting systematically the zero point of luminosity and color calibrations.

9. Explanation of local velocity distribution (Oort < 1928) using a simple kinematic model based on LSR relative to an idealized, circular orbit around the Galactic center.

In other words, we can now begin to build a simple model for the Galaxy based on dynamics, so let's do it.

We start with the coordinate system (r, φ, z) and the Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2) - \Phi$$

and assume a thin disk approximation. Thus from the Euler–Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) = \frac{\partial \mathcal{L}}{\partial x_i}, \quad \Phi = \Phi(r, z)$$

$$\frac{d}{dt}\dot{r} = r\dot{\varphi}^2 - \frac{\partial \Phi}{\partial r}$$

$$\frac{d}{dt}(r^2\dot{\varphi}) = 0$$

$$\frac{d}{dt}\dot{z} = -\frac{\partial \Phi}{\partial z}$$

From the last equation, we assert that $\left(\frac{\partial \Phi}{\partial z}\right)_{r_0, z=0} = 0$ so the motion in z is a simple harmonic oscillator at the solar distance (r_0):

$$\ddot{z} \simeq -\left(\frac{\partial^2 \Phi}{\partial z^2}\right)_{(r_0, 0)} z$$

with frequency $(|\Phi_0''|)^{1/2}$. Now if we call $\frac{d}{dt} \to v_z \frac{\partial}{\partial z}$, then:

$$W\frac{\partial W}{\partial z} = \left(\frac{\partial K_z}{\partial z}\right)_{(r_0, 0)} z$$

force (actually the acceleration ...)

is the gradient of the force in the vertical direction (locally) to the plane of the Galaxy, and $v_z \equiv W$ (the direction I defined in class). Again, we'll return to this form shortly but for now it is enough to note that the oscillation frequency in the direction $\hat{z}$ is $\omega_z^2 = |\Phi_{zz}|$ where $\Phi_{zz} \equiv (\partial^2\Phi/\partial z^2)_{(r_0, 0)}$.

Since $\partial\Phi/\partial\varphi \to 0$ by assumption

$$j = r^2\dot{\varphi} = \text{const} \quad \text{(again by assertion!)}$$

so we have:

$$\ddot{r} = \frac{j^2}{r^3} - \frac{\partial \Phi}{\partial r} \to \frac{j^2}{r_0^3} = \left(\frac{\partial \Phi}{\partial r}\right)_{r_0}$$

is the equation of motion for the reference circular orbit at r_0. OK, you know this one is a trivial solution for $\Phi \sim r^{-1}$, the Kepler problem, but for $\Phi(r, z)$ in general it is more complicated and we have no advance expectation that the Galactic mass is as centrally concentrated as a point mass. Then,

$$\Phi(r, z) \to \bar{\Phi}(r) \simeq \Phi_0 + \Phi_0' \delta r + \frac{1}{2}(\Phi_0'')\delta r$$

is a plausible expansion in the plane – assuming separability of Φ – and now taking $\Phi^{(n)} = \partial^n\Phi/\partial r^n$ with $\delta r = r - r_0$. So:

$$\ddot{\delta r} = -\left[\frac{1}{r^3}\left(\frac{\partial}{\partial r} j^2\right)\right]_0 \delta r \equiv -\kappa^2 \delta r$$

is the equation for the radial oscillation of the orbit, where

$$\kappa^2 \equiv \frac{1}{r^3}\frac{\partial}{\partial r}(r^4\dot{\varphi}^2) = \frac{1}{r^3}\frac{\partial j^2}{\partial r}$$

is called the <u>epicyclic frequency</u>.

Notice that if $\kappa^2 < 0$ the radial oscillations are not stable, hence:

$$\frac{\partial j}{\partial r} > 0$$

is a condition for generalized potentials; this is also called the Rayleigh criterion for any circulation (NB: for Keplerian motion, $j \sim r^{1/2}$; in general, what does this imply for $\omega \sim r^n$?).

Now assume we're on a platform moving at $\dot{\varphi}_0 = \omega$. Then we notice two things immediately: a Coriolis term appears naturally in the equations of motion and the local frequency is the same in δr and $\delta\varphi$ and is given by κ. In addition, note that κ is given by Φ_0'' which is the tidal potential! We obtain the same result from a simple perturbation expansion around Ω_0, the circular frequency of the LSR, using:

$$\dot{\varphi} \rightarrow \dot{\varphi} + \Omega_0 \quad : 2 r_0 \Omega_0 \delta r + r_0^2 \dot{\varphi} = 0$$

Now you also could figure much of this out, for any generalized mass distribution, from:

$$\Phi(r,\varphi,z) \xrightarrow[\partial_\varphi \to 0]{} \Phi(r,z) \xrightarrow[z=0]{} \Phi_0 + \Phi_0' \delta r + \frac{1}{2}\Phi_0''(\delta r)^2$$

is locally – always – around an equilibrium point $(r_0, z=0)$ we have a harmonic potential whose orbits are stable provided:

$$\Phi_0' = 0 \rightarrow r = r_0$$

$$\Phi_0'' \geq 0 \longrightarrow \omega^2 \equiv \kappa^2 = \Phi_0'' \quad \text{(stable only if a local minimum)}$$

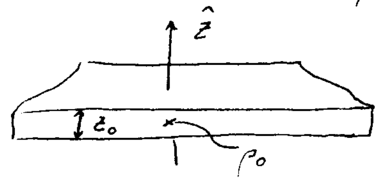
The anharmonic terms, those that ultimately really dominate the orbit of a star, are those for which Φ_0''' and $\Phi_0^{(iv)}$ don't vanish. For the gravitational case:

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} = -4\pi G\rho(r) \quad \text{but here we use only ordinary derivatives}$$

as before so using $\left(\frac{\partial \Phi}{\partial r}\right)_0 = \frac{G\mathcal{M}(r)}{r^2} - \kappa^2 = \frac{\partial}{\partial r}\frac{\mathcal{M}(r)}{r^2} \rightarrow \frac{\partial}{\partial r}[\Sigma(r)]$ where Σ is the surface density ~~distribution~~ (note $\frac{\partial}{\partial r} r k_r = 4\pi G\Sigma$).

NB To reiterate a point from class, the epicyclic frequency is the characteristic response of the system – the resonant frequency. It varies with distance from the galactic center. Since the stellar "gas" is collisionless, any resonant driving is reversible in the sense that once the perturbation, if present initially, is turned off the stars relax collectively back to their original state as in Landau damping.

Problem



Consider an infinite, self-gravitating planar layer in hydrostatic equilibrium. We'll say, for the moment, that the layer is stellar only, but it doesn't matter – all we need to know is that we don't require axial symmetry or other dynamical conditions. Choose an isothermal equation of state:

$$P = \rho\sigma^2 \quad , \quad \sigma = \sigma_w$$

Then:

$$\frac{1}{\rho}\frac{dP}{dz} = \frac{d\Phi}{dz}, \quad \text{(thin disk)}$$

NB in this last step, we're assuming $\Phi(r,z) \to \Phi(r_0, z)$ so $\partial_z \to d_z$. In general, for any arbitrary geometry, of course, this won't be true but if we confine $z \ll r_0$ it will hold. If we assert that σ = constant,

$$\frac{d\ln\rho}{dz} = \frac{d\Phi}{dz} \to \frac{d}{dz}\left(\frac{d\ln\rho}{dz}\right) = \frac{1}{\sigma^2}\frac{d^2\Phi}{dz^2} = -\frac{4\pi G}{\sigma^2}\rho \equiv -\frac{4\pi G\rho_0}{\sigma^2}\left(\frac{\rho}{\rho_0}\right)$$

the last step coming from $\nabla^2\Phi \Rightarrow \frac{d^2\Phi}{dz^2}$ and the Poisson equation. Now rescale the length. For this, we define:

$$z_0^{-1} = \left(\frac{4\pi G\rho_0}{\sigma^2}\right)^{1/2};$$

this quantity is the inverse of a characteristic length, we'll call it the density scale height. Since we are assuming hydrostatic equilibrium in $\hat{z}$, the vertical oscillations are statistically confined to this traverse. It is, in fact, the statement of why the disk looks thick: we have a collisionless but self-gravitating layer in which the stars are distributed with some $f(W)$ such that only a small fraction have random velocities much greater than σ. Hence, looking at an instant in time, most of the stars will be within z_0 of the midplane, with a few outliers. Note that

$$t_{ff}^2 \simeq \frac{1}{4\pi G\rho_0}$$

so you're assuming that any group having σ_i reaches a different scale height. Then, if we define:

$$y \equiv \ln\frac{\rho}{\rho_0} \qquad x \equiv \frac{z}{z_0}$$

we obtain:

$$\frac{d^2y}{dx^2} = -e^y,$$

the equation of structure for an isothermal slab.

Now for the hint for the solution:

$$y'y'' = -e^y y' \to \frac{1}{2}\frac{d}{dx}(y')^2 = -\frac{d}{dz}(e^y)$$

$$\to \frac{1}{2}(y'^2 - y_0'^2) = -(e^y - e^{y_0})$$

where $y_0 = 0$, what is y_0'? The rest you can solve again by integration using:

$$\bar{x} = \sqrt{2}\,x,$$

$$i\bar{x} = \int \frac{dy\, e^{-y/2}}{(1-e^{-y})^{1/2}} = \int \frac{du}{(1-u^2)^{1/2}}, \qquad u = \sin\xi \to \xi = \sin^{-1}u$$

and now I'll leave the rest for you.

 27/02/04 (5)

The obvious next step is to build a model isothermal sphere – in the stellar "gas" case, this is equivalent to two assertions about the equation for the "pressure":

1. the orbits are either ~~isotropic~~ radial or J, the total specific angular momentum, vanishes and all orbits and their phases are randomly distributed;
2. whatever we have for the population, it is homogeneous and constant mass with no differential cooling.

The first is spherical symmetry, the second the assertion of uniqueness for the velocity distribution function. Then:

$$\frac{1}{\rho}\frac{dP}{dr} = -\frac{GM}{r^2} = \frac{d\Phi}{dr} \longrightarrow \frac{d\ln\rho}{dr} = \sigma^2\frac{d\Phi}{dr}$$

and from the full radial Poisson equation we again have:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{1}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

(now a radial length scale)

$$\longrightarrow \frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) = e^{-y}, \quad y = -\ln\left(\frac{\rho}{\rho_0}\right), \quad x = \frac{r}{r_0}, \quad r_0 = \frac{\sigma}{(4\pi G\rho_0)}$$

$$y(0) = 0, \quad y'(0) = 0$$

First obtained by Emden for a stellar interior model, this is the isothermal version of the more general Lane-Emden equation (which really applies to $P = K\rho^\gamma$ for a generalized polytropic equation of state). This equation, alas, has no analytic closed form solution but $\ln\rho$ acts like Φ as you see that

$$m(x) \sim -x^2\frac{dy}{dx}$$

is the mass internal to x. This is the model we can use for the extended structure of the system, the halo. The matched solutions, with increasing J as we go to smaller radius, will produce the disk, the origin of which clearly requires some sort of cooling to get $\sigma/\Theta_0 \ll 1$.

NB: It suffices ~~that~~ in disk galaxies to say there are no blue (massive) halo stars. These are confined to the disk (thick or thin) and consequently all star formation ceased there long ago. But also obviously, at an earlier epoch one expects the halo to have been brighter and a more dominant part of the system with look-back time (evolutionary correction or galactic arrow of time). As I mentioned in class, the metallicity of the halo is also lower than the disk.