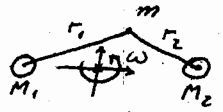
A note on the distribution function for the disks

28/2/04 (6)

For a self-gravitating slab, the vertical structure is given locally by $z_0 \sim \sigma t_{ff} \sim \frac{\sigma}{\rho_0^{1/2}}$ as we've seen. When σ is small, the self gravity of the disk dominates the structure and, depending on the radial variation of ρ_0, you'll see a radial dependence of z_0. If, however, the disk density is small relative to the central mass, then $z_0 \sim \frac{\sigma}{v_\varphi} r \sim r^{3/2}\sigma$. A cold ($\sigma$ small) disk is, therefore flaring but always thin. This is close to the case for planetary rings, or for the distribution of small bodies in the solar system – the asteroid belt. Now for the latter, resonances appear, not densely, with almost all planetary periods and the same is true for planetary rings and moon systems. There's a limit, however, to the closest spacing of resonances: when the spacing $\Delta r < \sigma \kappa^{-1}$, where κ is the epicyclic frequency, as before, the disks are effectively viscous and the resonances should overlap. Thus for "hot" systems, those with large σ, both the scale height and the smoothness of the disks should increase. In effect, the presence of ringlet structure ~~als~~ in planetary rings indicates that the disks must be cold and quite different from galactic disks – even if the galactic rings are not self-gravitating. You see such complex structures in shell galaxies and also polar rings, both produced by tidal interactions. Stars are injected into new orbits from cold disks (and therefore have high j relative to the case for disrupted halos or elliptical galaxies). Note also that if we take $\Sigma \sim \rho_0 z_0$ then we have $\sigma^{1/2}\kappa^{1/2}/\Sigma^{1/2}$ or $Q = \kappa\sigma/\Sigma$ ~~as~~ as a measure of the self-gravitation of a region of the disk. We'll return to this shortly, it's an important stability parameter.



For the next level of the dynamics, let me digress to a very different configuration – the classical three body problem (TBP). Now we have a specific form for the potential, known a priori. We put ourselves in the corotating frame and write the eqs. of motion in rectangular coordinates relative to the CM:

$$\ddot{x} - 2\omega\dot{y} = x\omega^2 - \frac{\partial\Phi}{\partial x}$$
$$\ddot{y} + 2\omega\dot{x} = y\omega^2 - \frac{\partial\Phi}{\partial y}$$

Here:

$$\Phi = -\frac{GM_1}{r_1} - \frac{GM_2}{r_2}$$

and:

$$\frac{\partial\Phi}{\partial x_i} = \frac{GM}{r_1^2}\frac{\partial r_1}{\partial x_i} + \frac{GM}{r_2^2}\frac{\partial r_2}{\partial x_i} \qquad \text{for } x_i = (x,y)$$

with distances from the CM $= (r_{01}, r_{02})$ and $r^2 = x^2 + y^2$. The assumptions required for solution are: circular orbit for (M_1, M_2) and $m \ll (M_1, M_2)$. In fact, while not always realized in the stellar case (binary system) for which the primary orbit may be eccentric (hence producing a periodic forcing on the orbital timescale), for a bar (extended mass distribution) stationarity of the potential is assured. More on this in a moment. For a binary, if the orbit is eccentric, m sees an oscillating dipole whose period is $\frac{2\pi}{\omega}$. Thus since for m the period may resonate with this timescale, but isn't necessarily the same, the angular momentum of the "test mass" may change. To continue: assume, somehow, you've already found $\{(x_0, y_0)\}$, the set of equilibrium points where $\ddot{x} = \ddot{y} = 0$. Then:

$$\Phi(x,y) \simeq \Phi_0 + \left(\frac{\partial\Phi}{\partial x}\right)_0 (x-x_0) + \left(\frac{\partial\Phi}{\partial y}\right)_0 (y-y_0)$$
$$+ \frac{1}{2}\left[\left(\frac{\partial^2\Phi}{\partial x^2}\right)_0 (x-x_0)^2 + 2\left(\frac{\partial^2\Phi}{\partial x\partial y}\right)_0 (x-x_0)(y-y_0)\right.$$
$$\left. + \left(\frac{\partial^2\Phi}{\partial y^2}\right)_0 (y-y_0)^2\right] + \cdots$$

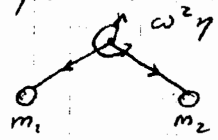
as the expansion around equilibria. Thus if we call $\xi = (x-x_0)$ and $\eta = (y-y_0)$ and assume:

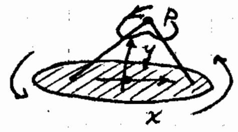
$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\xi_0\\ \eta_0\end{pmatrix} e^{i\sigma t}$$

we obtain:

$$-\sigma^2\xi - 2i\sigma\omega\eta = \omega^2\xi + \Phi_{xx}\xi - \Phi_{xy}\eta$$
$$-\sigma^2\eta + 2i\sigma\omega\xi = \omega^2\eta - \Phi_{xy}\eta - \Phi_{yy}\eta$$
$$\longrightarrow \quad (\sigma^2+\omega^2-\Phi_{xx})(\sigma^2+\omega^2-\Phi_{yy}) - (\Phi_{xy}^2 + 4\omega^2\sigma^2) = 0$$

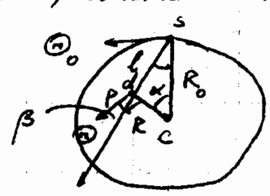
which, you see, is a quartic equation for $\sigma(\omega)$ (note, for the non-rotating case, that we have the normal harmonic oscillator solution in the (x,y) directions separately where $\sigma = \Phi_{xx}^{1/2}$ and $\sigma = \Phi_{yy}^{1/2}$ ~~for~~ only if $\Phi_{xy} = 0$. Otherwise we have coupled orbits. It's straightforward to see that if $\omega = 0$ for (m_1, m_2) the system is unstable). Why, you may wonder, are there two stable points not along the line connecting (m_1, m_2)? Simply, when at (x_0, y_0) as shown, the Coriolis force produces deflection <u>around</u> this equilibrium while the centrifugal potential balances the attraction of the two masses otherwise. Now if you have a bar whose rotation frequency is Ω, you've – in essence – taken the two masses (m_1, m_2) and made them a continuous distribution. The point <u>P</u> is again the locus at which the test mass can execute a stable orbit, will <u>still</u> exist and the points <u>along</u> the axis of the bar (if symmetric) will exist and be equally unstable as in the binary star case (L_2, L_3 – the Lagrangian points). But now, depending on the shape of the bar, the orbit may have a very distorted shape (the effect of Φ_{xy} for the bar) due to the precise relation between Φ_{xy} and either component of the Poisson equation (Φ_{xx} and Φ_{yy}). It's a separate question to seek the <u>origin</u> of a bar; that's a dynamical question.





Kinematic local treatment

The first model for streaming was, in fact, not dynamical. Instead, consider motion relative to the LSR, assumed to be moving at velocity $\Theta_0 \hat{\varphi}$ in the (r, φ, z) coordinate system. Take $(\sigma_u^2 + \sigma_v^2)^{1/2} \ll \Theta_0$, the cold disk approximation, and then if d is the distance at a galactic longitude l measured from S (solar distance R_0 from the Galactic center C) to a star P at distance R from the center, we can use simple geometry to obtain the radial and transverse (proper motion) velocity components and, from these, to construct the rotation curve for the disk. First:



$$\frac{\sin\alpha}{d} = \frac{\sin l}{R}, \qquad R\cos\alpha + d\cos l = R_0$$

and then for v_r we have

$$v_r = \Theta\cos\beta - \Theta\sin l = \Theta\sin(\alpha + l) - \Theta_0 \sin l$$

$$\Theta(R) = \Theta_0(R_0) + \left(\frac{d\Theta}{dR}\right)_0 (R - R_0) = \Theta_0(R_0) - \left(\frac{d\Theta}{dR}\right)_0 d\cos l$$

$$v_r = \left(\frac{\Theta}{R} - \frac{\Theta_0}{R_0}\right) R_0 \sin l$$

$$= \frac{1}{2}\left(\frac{\Theta_0}{R_0} - \left(\frac{d\Theta}{dR}\right)_0\right) d \sin 2l \equiv A\, d\sin 2l,$$

where A is called one of the two Oort constants. Note that we're assuming d is small relative to R_0, an important assumption. For the transverse velocity:

$$v_T = -\Theta\cos(l+\alpha) - \Theta_0\cos l$$

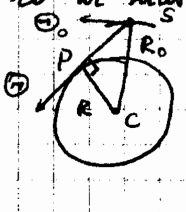
$$= \left(\frac{\Theta}{R} - \frac{\Theta_0}{R_0}\right) R_0 \cos l - \Theta\frac{d}{R}$$

$$= -A\cos 2l + -\frac{1}{2}\left(\frac{\Theta_0}{R_0} + \left(\frac{d\Theta}{dR}\right)_0\right) = -dA\cos 2l + B,$$

defining the second necessary constant. Note that the distance, d, must come from parallax observations which, for direct geometric measurements, limits the distances to roughly the solar neighborhood, around a few hundred pc. But now we have a problem! The disk is cold, but not that cold: $\sigma/\Theta_0 \approx 0.07$ so for any rotation curve $\Theta(R)$, we need:

$$\left(\frac{d\Theta}{dR}\right) d \gtrsim \sigma \qquad \left(\begin{array}{l} \gtrsim 8\times10^{-2} \text{ for } d \sim 200\,\text{pc} \\ \gtrsim 75\ \text{km s}^{-1}\,\text{kpc}^{-1} \end{array}\right)$$

Now for $(\Theta/R)_0$ we have (LSR ~ 220 km s^{-1}) ~ 25 km s^{-1} kpc^{-1} so we must look at the gradients to obtain the rotation curve. Further progress is made by looking at the tangent points, for which at a distance d we reach the maximum velocity along the line of sight, best achieved not with the stars but with the gaseous (HI) component.



So what do κ and (A, B) actually provide? If we have rigid rotation, then the local shear vanishes and the local epicyclic frequency is $2\Omega_0$, the rotation frequency for a rigid system - given by the Coriolis term (recall that, in rectilinear coordinates, $\ddot{x} - 2\Omega_0 \times \dot{x} = 0$ gives $\omega = 2\Omega_0$). If, however, we have differential rotation, then the shear (due to the tide from the mass distribution) provides an additional contribution - this is measured by the Oort constants.

With this material in hand, having the concepts of large scale structure and a velocity dispersion, we can now write the equations of motion for an ensemble of stars. We'd written $\mathcal{L}$ using only one star as a reference. But the equations of motion hold as well for a "gas" of stars, having a distribution function $f(v)$ for the random component around any possible mean motion.

Thus, let's go to the equation of evolution for $f(v)$ neglecting collisions:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + v_j \frac{\partial f}{\partial x_j} + \dot{v}_j \frac{\partial f}{\partial v_j} = 0 ,$$

the Vlasov equation is well known from plasma physics. It's actually trivial in this form, just the statement that $f \to f(t, \underset{\sim}{x}, \underset{\sim}{v})$ and saying that – if f doesn't evolve in form with time – then f has characteristics that are the dynamically conserved quantities. Now, as usual,

$$n = \int f \, d\underset{\sim}{v}$$

is the (normalizing) density and

$$n \langle v_i \rangle = \int f v_i \, d\underset{\sim}{v} , \quad n \langle v_i v_j \rangle = \int f(v) v_i v_j \, d\underset{\sim}{v}, \text{ etc.}$$

For convenience in the discussion, we'll define a tensor T_{ij} for the kinetic energies:

$$\underset{\approx}{T} = T_{ij} \hat{e}^i \hat{e}^j = n(V_i V_j + \langle \tilde{v}_i \tilde{v}_j \rangle) \hat{e}^i \hat{e}^j$$

where $\tilde{v}_i$ is the random component of $v_i = V_i + \tilde{v}_i$ such that $\langle \tilde{v}_i \rangle = 0$, and $\hat{e}^k$ are unit vectors. In a gas, we usually have:

$$\langle \tilde{v}_i \tilde{v}_j \rangle = \sigma_j^2 \delta_{ij}$$

if we assume that the f is gaussian, but now we don't have that luxury. This is why I stressed the importance of the form of the Schwarzschild distribution: by taking $\sigma_i \neq \sigma_j$ for any two (i,j) but still assuming gaussian statistics, we've actually imposed a behavior on the gas of stars that ignores possible correlated motions.

In other terms,

$$\langle v_i v_j \rangle = \langle (V_i + \tilde{v}_i)(V_j + \tilde{v}_j) \rangle = V_i V_j + \langle \tilde{v}_i \rangle V_j + \langle \tilde{v}_j \rangle V_i + \langle \tilde{v}_i \tilde{v}_j \rangle ,$$

but:

$$\begin{aligned} \langle v_i v_j v_k \rangle &= \langle (V_i + \tilde{v}_i)(V_j + \tilde{v}_j)(V_k + \tilde{v}_k) \rangle \\ &= V_i V_j V_k + \langle \tilde{v}_i \tilde{v}_j \tilde{v}_k \rangle + \langle \tilde{v}_i \tilde{v}_j \rangle V_k + \cdots + \langle \tilde{v}_i \rangle V_j V_k + \cdots \\ &\neq V_i V_j V_k + \langle \tilde{v}_i \tilde{v}_j \rangle V_k + \cdots \\ &\neq V_i V_j V_k + \sigma_j^2 \delta_{ij} V_k + \cdots \end{aligned}$$

so the closure problem is made more complicated. So taking the first moment with $v_i \hat{e}^i$:

$$\frac{\partial}{\partial t}(n V_i \hat{e}^i) + \frac{\partial T_{ij}}{\partial x_j} \hat{e}^i + T_{ij} \frac{\partial \hat{e}^i}{\partial x_j} = n K_i \hat{e}^i$$

NB Here we note that $\nabla \cdot \underset{\approx}{T} = \hat{e}^j \frac{\partial}{\partial x_j}(T_{ij} \hat{e}^i)$ and $T_{ij} \hat{e}^i$ is a vector (sum over repeated indices).

where we've defined K_i to be the acceleration component. For a Cartesian coordinate system, $\hat{e}^i$ is independent of the x_i, but not for our disk system:

$$\frac{\partial \hat{\varphi}}{\partial \varphi} = -\hat{r} , \quad \frac{\partial \hat{r}}{\partial \varphi} = \hat{\varphi}$$

and:

$$\frac{\partial Q}{\partial x_j} \to \frac{1}{r}\frac{\partial}{\partial r} rQ + \frac{1}{r}\frac{\partial Q}{\partial \varphi} .$$

It's easy to see that $T_{ij} = T_{ji}$ and we have only $(T_{rr}, T_{\varphi\varphi}, T_{zz}, T_{r\varphi}, T_{rz}, T_{\varphi z})$ (6 independent components) so that :

$$\frac{\partial}{\partial t}(nV_r) + \frac{1}{r}\frac{\partial}{\partial r} r T_{rr} + \frac{T_{rr} - T_{\varphi\varphi}}{r} + \frac{\partial}{\partial z} T_{rz} = n k_r$$

$$\frac{\partial}{\partial t}(nV_\varphi) + \frac{1}{r}\frac{\partial}{\partial r} r T_{r\varphi} + \frac{2}{r} T_{r\varphi} + \frac{\partial}{\partial z} T_{z\varphi} = n k_\varphi$$

$$\frac{\partial}{\partial t}(nV_z) + \frac{1}{r}\frac{\partial}{\partial r}(r T_{rz}) + \frac{\partial}{\partial z} T_{zz} = n k_z$$

$$\frac{1}{r}\frac{\partial}{\partial r} r k_r + \frac{1}{r}\frac{\partial}{\partial \varphi} k_\varphi + \frac{\partial}{\partial z} k_z = -4\pi G n$$

where the last term is the Poisson equation in less-than-familiar form. We'll impose two conditions:

$$\frac{\partial}{\partial t} \to 0, \quad \frac{\partial}{\partial \varphi} \to 0 \; : \; k_\varphi \to 0 .$$

Then we'll assume:

$$V_{\varphi,0}^2 = -r K_r .$$

Now we have $\sigma_{rr} \equiv \langle \tilde{v}_r^2 \rangle$, $\sigma_{r\varphi} = \langle \tilde{v}_r \tilde{v}_\varphi \rangle$, $\cdots$, so in steady state and if the system is in orbital motion only (no radial streaming, a natural consequence of the imposed symmetry), then:

$$\frac{1}{r}\frac{\partial}{\partial r}(r n \sigma_{rr}) + \frac{n(\sigma_{rr} - \sigma_{\varphi\varphi})}{r} - n\frac{V_\varphi^2}{r} + \cancel{\frac{\partial}{\partial z}(n\sigma_{rz})} = -n\frac{V_{\varphi 0}^2}{r}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r n \sigma_{r\varphi}) + \frac{2}{r} n \sigma_{r\varphi} + \cancel{\frac{\partial}{\partial z}(n \sigma_{z\varphi})} = 0$$

(further assuming planar symmetry) we have

$$(V_{\varphi 0}^2 - V_\varphi^2) = (\sigma_{\varphi\varphi} - \sigma_{rr}) - \frac{1}{n}\frac{\partial}{\partial r}(n r \sigma_{rr}),$$

the equation for the vertex deviation. Notice that the departure of V_φ from the LSR, which we assumed to be $V_{\varphi,0}$, comes from only diagonal terms in the velocity dispersion, σ_{rr} and $\sigma_{\varphi\varphi}$; the last term is the population-weighted value of the gradient in this dispersion, and even for a gaussian ($\sigma_{rr} = \sigma_{\varphi\varphi}$, for instance) we still see $V_\varphi \neq V_{\varphi,0}$.

These equations form the basis of "stellar hydrodynamics", and are the close connection between plasmas and stars. Note that we do not find f from these. In fact, we ignore it: it enters only as we average over the distribution of all velocities. Then we can go one step further. Since we know that the hydrodynamic equations written in non-conservative form give us the virial theorem when taking moments over space, it's clear (I hope) that the same product gives the analogous result for the stellar equations;

$$x_i \hat{e}^i \frac{\partial}{\partial t}(\rho V_i)\hat{e}^i = x_i \frac{\partial}{\partial t}\rho V_i = \frac{\partial}{\partial t}(\rho V_i x_i) - \rho V_i \dot{x}_i \to -2\rho V_i V_i$$

if we take a steady state globally (that is $\partial_t(\rho V_i x_i) = \partial_t(\rho \underset{\sim}{V} \cdot \underset{\sim}{x}) = 0$, we'll return later to this assumption); but also

$$x_i \hat{e}^i \frac{\partial}{\partial x_j}(T_{ij}\hat{e}^i) = x_i \frac{\partial}{\partial x_j} T_{ij} = \frac{\partial}{\partial x_j}(T_{ij} x_i) - T_{ij}\delta_{ij} \quad \therefore$$

$$-2\rho V_m^2 - \bar{T}_{ij}\delta_{ij} = \rho x_i k_i m$$

where $\bar{T}_{ij} = \int d\underset{\sim}{x}\, T_{ij}$ ($d\underset{\sim}{x}$ is a volume element) and $m = \int \rho\, d\underset{\sim}{x}$. The divergence term vanished, as usual, because of the imposed constraint that our system has no walls (no surface terms). Now we have:

$$\bar{T}_{ij}\,\delta_{ij} = \mathrm{Tr}(\bar{T}_{ij})$$

but this is not the same as the pressure, it only looks the same. In the case – the ideal gas with a gaussian (Maxwellian) – of an isotropic velocity dispersion, $\bar{T}_{ij}\,\delta_{ij} = \int 3P\,dV$. Now we simply have $\int \rho(\sigma_{rr}+\sigma_{\varphi\varphi}+\sigma_{zz})\,d\underset{\sim}{x}$. The last term is the potential energy, the first is the kinetic energy, so fundamentally we've recovered the classical virial theorem, as we must:

$$2T + \Omega = 0 \qquad (\text{here } T = \tfrac{1}{2}\int \rho V^2 d\underset{\sim}{x}\text{, the kinetic energy})$$

$\left(K_i = -\dfrac{\partial \Phi}{\partial x_i} = \Phi\,\dfrac{x_i - x_{io}}{r}\right)$. There's nothing of great surprise here; after all, we've been assuming collisionless steady state. But the importance is that any stellar system has a finite binding energy, as with any self-gravitating body, and therefore has an escape velocity given by:

$$E_{tot} = T + \Omega = 0 \Leftrightarrow E_{tot} = -\tfrac{1}{2}\Omega)$$

Now the greater the velocity dispersion, the more stars lie above the critical energy T_*. In other words, each star that leaves the system carries away binding energy and causes the system to contract. You've already encountered this idea before: the Kelvin-Helmholtz contraction for a self-gravitating star. Depending on the locus of mass loss, however, the whole system or only a part of it may contract. If mass is lost only from the surface – like photons from a star that remains otherwise in thermal equilibrium, the whole mass contracts. If, in contrast, the core alone loses mass, the envelope will expand – just as you see on the red giant branch – with the result that a homologous structure evolves to a very compact core plus extended halo.

NB This is most important for globular clusters, not galaxies as a whole. It requires collisional redistribution of momentum since it must be density dependent (in effect, the gravitational interaction rate depends on the density and the thermal timescale acts like the gravitational interaction timescale). You actually see this. Shapley (1930s) distinguished classes of clusters (based on their degree of central concentration), with our picture now being that the most concentrated are completely relaxed, but may still not be isotropic.

NB We note here the importance of substructure in the stellar system – the higher the binding energy the greater the velocity dispersion. Thus, stars lost to what I've called the "field", which formed in clusters, "cool" the clusters while increasing the dispersion of the field, analogous to the effect of photoionization on a gas.