In the gravito-acoustic instability, you're faced with a self-gravitating sound wave (as I had discussed in class) that has a restoring force due to its own gravity. In a gravity wave, we see something simpler - instead of a back-reaction on the gravitational field, the material simply responds buoyantly to $g \cdot \hat{z}$, the background gravitational acceleration. This is also true for a galaxy - for a dark matter halo - since the halo masses are so large the potential doesn't respond to changes in the disk. The fluctuations of the luminous matter are, therefore, only locally important. The potential in which the disk is imbedded remains rigid.

For the disk, on the other hand, the mean density ρ_0 may be locally larger - and vary on smaller scale, - than the halo (whose scale length is always larger) - so we can form very compact structures locally. Now let's see how. Start with:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} \rho v_j = 0$$

$$\rho\left(\frac{\partial}{\partial t} + v_j \frac{\partial}{\partial x_j}\right) v_i = -\frac{\partial P}{\partial x_i} + \rho \frac{\partial \Phi}{\partial x_i}$$

$$\frac{\partial^2}{\partial x_j \partial x_j} \Phi = -4\pi G \rho;$$

the usual fluid equations, supplemented by the Poisson (field) equation. This last is needed because of the feedback: changes in ρ produce changes in $\partial_x \Phi$. To be precise:

$$\partial_t \delta\rho + \delta v_j \frac{\partial \rho_0}{\partial x_j} + v_{0j} \frac{\partial}{\partial x_j} \delta\rho + \rho_0 \frac{\partial}{\partial x_j} \delta v_j + \delta\rho \frac{\partial}{\partial x_j} v_{0j} = 0$$

reduces, for a uniform background and $v_{0j} = 0$ to:

$$\frac{\partial}{\partial t} \delta\rho + \rho_0 \frac{\partial}{\partial x_j} \delta v_j = 0 .$$

I'll note here something else from class:

if the density, ρ_0, of the background is changing systematically with time, as we know from the cosmic expansion, an additional term appears, $-(3\dot{a}/a)\rho_0(0)$, since $\rho_0(t) = \rho_0(0)\left(\frac{a(t)}{a(0)}\right)^{-3}$ for some scaling parameter for distances; we'll discuss soon the consequences of this.

Having assumed $v_{0j} = 0$ and $\rho_0 =$ constant, we choose a barotropic equation of state, one for which $P \to P(\rho)$ only so pressure surfaces are equipotentials and we explicitly ignore thermal instabilities:

$$\delta P = \left(\frac{\partial P}{\partial \rho}\right)\delta\rho = \underbrace{\sigma^2}_{\text{(compressibility)}}\delta\rho .$$

Never mind that you don't know what σ^2 is, we'll assume for now that it's some internal random motion and independent of ρ. Then

$$\rho_0 \frac{\partial \delta v_i}{\partial t} = -\sigma^2 \frac{\partial \delta\rho}{\partial x_i} + \rho_0 \frac{\partial \delta\Phi}{\partial x_i} ;$$

but then from:

$$\frac{\partial^2}{\partial x_j \partial x_j} \delta\Phi = -4\pi G \delta\rho$$

$$\rightarrow \quad \rho_0 \frac{\partial}{\partial t}\underbrace{\left(\frac{\partial}{\partial x_i}\delta v_i\right)}_{-\frac{1}{\rho_0}\frac{\partial}{\partial t}\delta\rho} = -\sigma^2 \frac{\partial^2}{\partial x^2}\delta\rho - 4\pi G \delta\rho \cdot \rho_0$$

yielding:

$$-\frac{\partial^2}{\partial t^2}\delta\rho = -\sigma^2 \frac{\partial^2}{\partial x^2}\delta\rho - (4\pi G\rho_0)\delta\rho .$$

This linear wave equation has a single mode solution:

$$\omega^2 - k^2\sigma^2 = -4\pi G\rho_0$$

with a critical wavenumber:

$$\omega^2 = 0 \rightarrow k_*^2 = \frac{4\pi G\rho_0}{\sigma^2},$$

First derived by Jeans, this was an estimate for what happens in the case of a 1D slab; but the generalization to 3D is relatively simple. However, the nonlinear effects are not so simple. When we include:

$$\delta v_j \frac{\partial}{\partial x_j} \delta v_i,$$

then we couple very large k_j, small length scales, to large scales – this term is a convolution taken over all wavenumbers. Including such terms, along with the velocity fluctuations from random motions, redistributes the motion among the various lengths and ultimately leads to turbulence (in fact, to self-gravitating turbulence). But now look at k_*. It depends on $(\sigma/\rho_0^{1/2})^{-1}$. Whatever σ is, and if it is due to turbulent motions, it's always large, if it is not due to density or temperature it may always be large. ~~But~~ Instead, consider now what happens as the structure (instability) grows. Collapse causes ρ_0 to increase. Now $M_J(t) < M_J(0)$, so substructure (fragmentation) must start! This is what I talked about in class: in Hoyle (1953) this was introduced to explain the formation of lower mass objects. This fragmentation continues unless σ stars to increase. If σ is c_s, the isothermal sound speed, then $\tau \sim \rho l \sim \rho^{1/2}$ increases as $\rho_0(t)$ increases. This – the optical depth – reduces the cooling rate. Thus there is a minimum mass, limited only by the rate of cooling, a point discussed for protostellar formation of primordial objects by Palla, Salpeter, and Stahler (1983) (note the very long time after Hoyle!).

NB If the rate of expansion of the medium, $-\frac{3\dot{a}}{a}\rho_0$, is fast enough, this stops the growth of the largest fragments. But since, as we'll see later, $\dot{a}/a \equiv H_0$, the Hubble parameter, the growth of structure comes in part from the competition between H_0^{-1} and $t_{ff} \sim \rho_0^{-1/2}$.

So for the instability, you have:

$$\delta\rho \rightarrow k\delta\Phi \rightarrow \rho_0 \delta v$$

as a schematic sequence. For the gravito-acoustic mode, if $\sigma = c_s$, we're asking for the rarefaction wave I discussed in class when $t_{acoustic} = l/c_s \approx t_{ff}$ so

$$l_* \sim \frac{c_s}{(G\rho_0)^{1/2}} \leftrightarrow \lambda_J \text{ (one usual symbol for Jeans length).}$$

Now we have another timescale if the medium is sheared, given by $t_{shear} \sim \kappa^{-1}$ (where, remember, κ is the epicyclic frequency). Then:

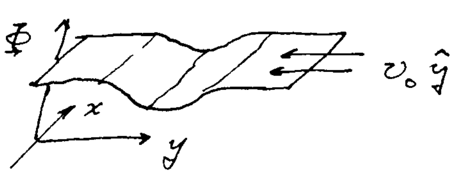
$$\frac{t_{shear}\, t_{acoustic}}{t_{ff}^2} \approx 1 \rightarrow \frac{lG\rho_0}{\kappa c_s} \approx 1 \rightarrow l_* \sim \frac{\kappa c_s}{G\rho_0}$$

(or for a slab, replace $\rho_0 l \rightarrow \Sigma_0$, the surface density,

$$Q \approx \frac{\kappa c_s}{G\Sigma_0} \rightarrow \text{Toomre parameter})$$

and you see the sound – or turbulent – crossing time combines with the shear to give an instability parameter first defined by Toomre for a sheared sheet. If $Q \simeq 1$, the layer is marginally stable, while if t_{shear} and/or $t_{acoustic}$ are/is less than t_{ff}, the instability doesn't grow.

The simple case we discussed of a line slab can be translated into a flow problem if we imagine an equipotential surface (using here $\hat{z} \leftrightarrows \hat{\Phi}$). Now we have δg changing sign at some y. Then the flow equations are:



$$v_y \frac{dv_y}{dy} = -\frac{1}{\rho}\frac{dP}{dy} - \rho g$$

$$\rho v_y = J = \text{constant}$$

and again take g given by $\delta\Phi$, some perturbation in the background potential. Now we can write:

$$\frac{1}{v_y}(v_y^2 - \sigma^2)\frac{dv_y}{dy} = -g$$

For a change in g, we must have an acceleration into the potential minimum so either $v_y = \sigma$ at $g = 0$ or $dv_y/dy = 0$ or both. But since dv_y/dy must also change sign when g does, the density must build up on the rear side of the potential minimum and therefore Q must also change ($\Sigma_0 \uparrow$, $Q \downarrow$). The result is a possible gravitational instability on the rear side, and subsequent collapse and star formation (or growth of a self-gravitating structure).