Now to include, as we began in class, the effect of a magnetic field. The compression of the field increases its energy density which, for a perfectly conducting, completely ionized medium, produces a back reaction that we can think of as a pressure. Under mass loading, this will generate a wave, analogously to an acoustic wave:

$$\frac{P}{\rho} \sim \frac{B_0^2}{\rho_0} \rightarrow v \sim \frac{B_0}{\rho_0^{1/2}}$$

In the case where B_0 is large, this may dominate the sound speed, resulting in an anisotropic coupling of the gas. Note that B_0 has a direction, so this motion is only transverse to the field but propagates <u>along</u> the field direction (NB since $\nabla\cdot\underset{\sim}{B} = 0$ always, and this is equally true for $\delta\underset{\sim}{B}$, such a wave cannot be compressible). This is an Alfvén wave, and it is a basic result of MHD theory. For an incompressible fluid, the equations for the field are:

$$\nabla\times\underset{\sim}{B} = \frac{4\pi}{c}\underset{\sim}{J} \rightarrow \nabla\times\delta\underset{\sim}{B} = \frac{4\pi}{c}\delta\underset{\sim}{J}$$

$$\partial_t\underset{\sim}{B} = \nabla\times\underset{\sim}{v}\times\underset{\sim}{B} \rightarrow B_0\nabla\times(\underset{\sim}{v}\times\hat{b})$$ (in effect, for single mode expansion, this serves the same role as the continuity equation for an acoustic signal)

This is different (!) from either Jeans or acoustic modes $\rightarrow$ $\nabla\cdot\underset{\sim}{B} = 0$

which couple to the equation of motion:

$$\rho\,\partial_t v = \frac{1}{c}\underbrace{\underset{\sim}{J}\times\underset{\sim}{B}}_{\to\,\delta\underset{\sim}{J}\times\underset{\sim}{B}_0} = \frac{1}{4\pi}(\nabla\times\underset{\sim}{B})\times\underset{\sim}{B} \longrightarrow \frac{\underset{\sim}{B}_0}{4\pi\rho_0}(\nabla\times\delta\underset{\sim}{B})\times\hat{b} = \frac{\partial v}{\partial t}$$

to give the dispersion relation

$$v_A^2 \equiv \frac{\omega^2}{k^2} = \frac{B_0^2}{4\pi\rho_0}$$ for 1D motion, $\underset{\sim}{B}_0 = B_0\hat{b}$ (for instance)

so we were only wrong by a factor of $\sqrt{4\pi}$ in our estimate (not bad!). But we made some <u>strong</u> assumptions: incompressibility ($\nabla\cdot\underset{\sim}{v} = 0$ results, and also $\nabla P \to 0$) and we've pre-linearized the equations ($\underset{\sim}{J}_0 = 0$, all terms involving gradients of $\underset{\sim}{B}_0$ vanish). But we have the basic form for the disturbance. For a case where the pressure is not neglected, we have two possible contributions, v_A and c_s. NB: in both cases, because we're assuming $\delta\rho/\rho_0 \ll 1$ and $|\delta\underset{\sim}{B}|/|\underset{\sim}{B}_0| \ll 1$, we obtain only a phase velocity (or, in other words, we have no lag between the group and phase velocities) and there is no net momentum or energy transfer to the medium. The waves are not dispersive, and they will not steepen to become shocks. Instead, we have a way for the medium to "radiate" if it is coupled at long range, so this is a loss mechanism.

<u>NB</u>: It's often said that Alfvén waves can supply a pressure support without being dissipative. Actually, this isn't correct - they do disperse in the large amplitude limit and also form shocks.

<u>Problem</u> Use single mode expansions with $\underset{\sim}{B}_0 = B_0\hat{b}$ and $P = \rho c_s^2$ to find a general solution for a magneto-acoustic dispersion relation in the linearized limit.

<u>Hint</u>: using this, you will have as the primitive equations:

$$-i\omega\delta\rho + \rho_0\, i\,\underset{\sim}{k}\cdot\underset{\sim}{v} = 0$$

$$-i\omega\delta\underset{\sim}{B} = i\,\underset{\sim}{k}\times(\underset{\sim}{v}\times\underset{\sim}{B}_0) \qquad \begin{pmatrix} \nabla\times \to i\underset{\sim}{k}\times \\ \partial_t \to -i\omega \\ \nabla \to i\underset{\sim}{k} \end{pmatrix}$$

$$i\,\underset{\sim}{k}\times\delta\underset{\sim}{B} = \frac{4\pi}{c}\delta\underset{\sim}{J}$$

$$-i\omega\rho_0\underset{\sim}{v} = \frac{1}{c}\delta\underset{\sim}{J}\times\underset{\sim}{B}_0 - i\underset{\sim}{k}c_s^2\delta\rho,$$ also $\nabla\cdot\delta B = \underset{\sim}{k}\cdot\delta\underset{\sim}{B} = 0$

noting that now we're using explicitly the conditions ρ_0 = const., $|\underset{\sim}{B}_0|$ = const, and $\underset{\sim}{J}_0 \overset{?}{=} 0$. You'll also notice that although $\underset{\sim}{k}\cdot\delta\underset{\sim}{B} = 0$, $\underset{\sim}{k}\cdot\underset{\sim}{v} \neq 0$, so there are now compressible (longitudinal) modes.

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Thus, with a magnetic field the medium has an additional pressure support. This requires complete coupling between the material and field components. But the gas has another possibility for the equation of state: it may be partially ionized. In other words, there can be a neutral phase that is not directly coupled to the $\underset{\sim}{B}$ field.

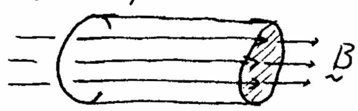
NB Actually, as we discussed in lecture, the neutral phase is not a separate fluid; it's the same particle in a different state. Consequently, a single particle spends some fraction of its time in a neutral state and can – in that state – move freely across magnetic field lines (I'm here using the usual expedient of fieldlines but, of course, these don't really exist). For a self-gravitating slab, for instance, this is an essential feature of the structure: the $\underset{\sim}{B}$ field adds to the support of the medium but may only delay, not cancel, the onset of gravitational instability. This is the process of ambipolar diffusion.

For a cloud with magnetic support:

$$\frac{1}{r}\frac{d}{dr}\left(\frac{r}{\rho}P'_{tot}\right) = -4\pi G\rho \qquad \left(P'_{tot} \equiv \frac{dP_{tot}}{dr}\right)$$

where $P_{tot} = \frac{1}{8\pi}B^2 + \rho c_s^2$. Now if $B = K\rho$, the condition for a frozen-in field, we have:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\rho}{dr}\right) = -\frac{(4\pi)^2 G}{K^2}\rho$$



which is essentially the same condition we found for the Lane-Emden equation, but not quite. First, notice the dimensions: we have $\frac{1}{r}\frac{d}{dr}r \leftrightarrow \nabla_r\cdot$ (for the radial component of the divergence). This is a cylinder (!). Then, you see that since $K = B_0/\rho_0$, the scale length is:

$$\lambda_M^{-1} = \frac{4\pi G^{1/2}}{K},$$

(NB: you can see easily how this arises from the same equations for MHD that gave us the Alfvén solution; prelinearize by setting $\underset{\sim}{B}\cdot\nabla\underset{\sim}{B}$ to zero and write the total pressure as $P_{mag} + P_{gas}$. Then take $P_{mag} \gg P_{gas}$ and the rest follows).

For the cylinder, we have another easy way of getting the stability limit – since the Alfvén wave crossing time is l/v_A then:

$$\frac{t_A}{t_{ff}} \sim \frac{l\rho_0^{1/2}}{B_0}\cdot(G\rho_0)^{1/2} \rightarrow l \sim \frac{B_0}{G^{1/2}\rho_0}$$

and we obtain one more very interesting result. Since $B_0 l^2 \approx \Phi$ is conserved, the mass is:

$$m_M \sim \Phi G^{-1/2}, \qquad (m_M \sim \rho_0 B_0 l^2 \rho_0^{-1} G^{-1/2})$$

actually independent of the density! This last may seem like a trick but it's not, and it doesn't depend at all on the precise geometry. Things change for the magnetoacoustic case, since then we must include c_s, but the same principle applies. Notice also that the effect of geometry is to modify the proportionality constant but nothing else. Now for the ambipolar diffusion case, several things change. First, B is no longer linearly proportional to ρ. The density changes as the field remains essentially unchanged. It may be true that a small change in the ionization can be neglected, the cross-field diffusion becomes very slow when $f \equiv n_p/n \simeq 1$; but if f is small, then only the collisional mean free path and local gravitational acceleration determines the diffusion rate. The drift velocity is $v_D \sim g t_c$ for $g \sim 2\pi G\Sigma$. Since $t_c^{-1} \sim n$, then v_D depends only on the scale height of the gas layer. A magnetic field modifies v_D by a factor of $(1+\omega_B^2 t_c^2)^{-1}$, where the Larmor frequency $\omega_B \sim \langle Z \rangle \sim f$ (Z is the charge).

Thus although the magnetic field slows the process, for a ~~com~~ partially ionized medium there is an inevitable drift and accumulation of mass in the central region of a self-gravitating cloud: $\mathbf{B}$ can slow but not prevent the collapse. This also addresses the fundamental problem of how to form stars even when a $\mathbf{B}$ field is present (first examined by Mestel and Spitzer in the '50s). The field is eventually "expelled" from the accumulating gas layer! You also see that since the motion $\mathbf{v}\cdot\mathbf{B}$ is unimpeded by the field, the configuration collapses dynamically in the $\hat{b}$ direction when $\Sigma > \Sigma_{crit}$ (given by the 1D solution for an isothermal disk) and then drifts inward in the radial direction ($\perp$ to $\hat{b}$). The resulting configuration may be force free [$(\nabla\times\mathbf{B})\times\mathbf{B} = 0 \to \nabla\times\mathbf{B} = \alpha\mathbf{B}$ with α a scalar function] but, in general, the exact solution is complicated (look in ADS under the heading "isopedic disks").