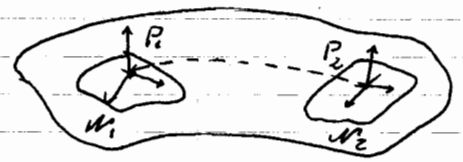
To justify the necessity to use general relativity for cosmological models, I'll remind you of the point I made today at the end of lecture. We sit at a fixed point, say $P_1$, in the universe and attempt to construct the physical laws from *local* measurements, within some neighborhood ($\mathcal{N}_1$) (or "coordinate patch"). Some other location, $P_2$, has observers who do the same thing in a neighborhood $\mathcal{N}_2$. If the laws are the same, so we can say there's *no* special location, then the two patches must be equivalent. We can then imagine a set of overlapping patches that permit us to move smoothly from one place to the other, not by a trajectory but with a continuous atlas of such patches. It doesn't matter how many dimensions we require for this space, using a 2D or 3D surface as here is just a convenience. Now we can construct the coordinate distance as a generalization of the flat space:




$$ds^2 = \sum_{ij} g_{ij}\, dx^i dx^j \Rightarrow g_{ij}\, dx^i dx^j \quad \text{(sum over repeated indices)}$$

for this local patch for tangent vectors $dx^i$, $dx^j$, the form we'll call *contravariant*. Then this is the analog of the scalar product in generalized coordinates. The quantity (matrix) $g_{ij}$ is the *metric*, the quantity that contains all the information about how to change from the primitive displacements $dx^i$ to the actual distance (in other words, if we move some number of steps $dx^i$, how "far" have we actually gone?). For example, on a plane, $dx^i = (dx, dy, dz)$ so

$$ds^2 = dx^2 + dy^2 + dz^2.$$

For a cylinder, to move an angular distance $d\varphi$ = constant, we have an arclength $dl = r\,d\varphi$ while $dr$ and $dz$ are linear:

$$ds^2 = dr^2 + r^2 d\varphi^2 + dz^2,$$

For a sphere, on $S^2$, we have $(d\vartheta, d\varphi)$ as angular (dimensionless) coordinates (I mean no "metrical" units) so:

$$ds^2 = dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2,$$

Ok, these are obvious and familiar. Then the displacements for the gradients are also given in $dl$, not $d\vartheta$, for instance ($d\vartheta$ along great circles, for instance, while $d\varphi$ is not). From these examples, you see that:

$$g_{ij} = g_{ji} \quad \text{(symmetry)}$$

and for these spaces, since they're only separably quadratic, $g_{ij} = g_i \delta_{ij}$ (diagonal), but in general they don't need to be. Actually, each of these is just the same as a flat space for $\mathcal{N}$ small enough (or if $r \to \infty$). So we have a geometrical device, one more thing needs to be added. To go from geometry to dynamics, we need some notion of *ordered* motion, something to tell us that for a sequence of locations $(P_1, P_2, \ldots)$ we have $P_1 \prec P_2 \prec P_3 \prec \cdots \prec P_n \prec \cdots$, where "$\prec$" means "before". Such an ordering we'll call – in classical terms, and later will generalize – a "temporal" sequence. Then we have a parameter, call it $\lambda$, such that $x^i(\lambda)$ is this ordered sequence. If we define the velocity as:

$$v^i = \frac{dx^i}{d\lambda}$$

then in our usual notation of scalar product giving the kinetic energy we have:

$$\widehat{T} = \frac{1}{2} g_{ij} v^i v^j = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j.$$

Ok, *now* we're ready. From *any* arclength $ds$, we can ask:

$$S = \int_{P_1}^{P_2} ds \;\overset{?}{\Rightarrow}\; \delta S = 0 \quad (\text{and } \delta^2 S < 0),$$

in other words, what is the minimum distance between $P_1$ and $P_2$? In fact,

This is the same (at fixed time) as asking what is the minimum for $\mathcal{T}$?. So:

$$\delta \mathcal{T} = \frac{\partial \mathcal{T}}{\partial x^i}\delta x^i + \frac{\partial \mathcal{T}}{\partial \dot{x}^i}\delta \dot{x}^i$$

$$= \frac{\partial \mathcal{T}}{\partial x^i} + \frac{d}{d\lambda}\left(\frac{\partial \mathcal{T}}{\partial \dot{x}^i}\delta x^i\right) - \delta x^i \frac{d}{d\lambda}\left(\frac{\partial \mathcal{T}}{\partial \dot{x}^i}\right)$$

noting that:

$$\delta\left(\frac{dx^i}{d\lambda}\right) = \frac{d}{d\lambda}\delta x^i.$$

Keeping $(P_1, P_2)$ fixed in both $\Delta\lambda$ and $x^j$ $(x^j(\lambda_1), x^j(\lambda_2))$, we then have:

$$\int_{P_1}^{P_2} \delta \mathcal{T} \rightarrow \frac{\partial \mathcal{T}}{\partial x^i} = \frac{d}{d\lambda}\left(\frac{\partial \mathcal{T}}{\partial \dot{x}^i}\right).$$

Since the variational is taken with respect to an arbitrary coordinate change, we'll write instead:

$$\frac{\partial \mathcal{T}}{\partial x^k} = \frac{d}{d\lambda}\left(\frac{\partial \mathcal{T}}{\partial \dot{x}^k}\right) \rightarrow \frac{\partial}{\partial x^k}(g_{ij}\dot{x}^i\dot{x}^j) = \frac{d}{d\lambda}\left(g_{ij}\left[\frac{\partial \dot{x}^i}{\partial \dot{x}^k}\right]\dot{x}^j + g_{ij}\left[\frac{\partial \dot{x}^j}{\partial \dot{x}^k}\dot{x}^i\right]\right).$$

Why? We're asserting that $g_{ij}$ depends only on $(x^i, x^j)$ and not on $(\dot{x}^i, \dot{x}^j)$. Now:

$$\frac{\partial \dot{x}^i}{\partial \dot{x}^k} = \delta^i_k = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \quad \text{Kronecker-}\delta$$

so that the second term is:

$$\Rightarrow \frac{\partial}{d\lambda}\left(g_{ij}\delta^i_k\dot{x}^j + g_{ij}\delta^j_k\dot{x}^i\right) = \frac{d}{d\lambda}\left(g_{ik}\dot{x}^i + g_{jk}\dot{x}^j\right)$$

Further:

$$\frac{d}{d\lambda}g_{ij} = \dot{x}^l\frac{\partial}{\partial x^l}g_{ij}.$$

So completing the expansion we have:

$$\frac{\partial g_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j = \frac{\partial g_{ik}}{\partial x^l}\dot{x}^l\dot{x}^i + \frac{\partial g_{jk}}{\partial x^l}\dot{x}^l\dot{x}^j + g_{jk}(\ddot{x}^j) + g_{ik}(\ddot{x}^i)$$

$$= \frac{\partial g_{jk}}{\partial x^l}\dot{x}^l\dot{x}^j + \frac{\partial g_{ik}}{\partial x^l}\dot{x}^l\dot{x}^i + 2g_{mk}\ddot{x}^m$$

Taking $g^{ij}$ to be the components of the inverse of $g_{ij}$, then:

$$\frac{1}{2}g^{mk}\left\{\left[\frac{\partial g_{ij}}{\partial x^k}\right]\dot{x}^i\dot{x}^j - \left[\frac{\partial g_{ik}}{\partial x^l}\dot{x}^l\dot{x}^i + \frac{\partial g_{jk}}{\partial x^l}\dot{x}^l\dot{x}^j\right]\right\} = \ddot{x}^m$$

giving:

$$\ddot{x}^m + \Gamma^m_{lk}\dot{x}^l\dot{x}^k = 0$$

where:

$$\Gamma^m_{lk} \equiv \text{Christoffel symbol} \equiv \text{connection};$$

you'll see soon how this is computed but for now it's enough to see how it formally enters the "equation of motion". I'm putting this in quotes because it appears, oddly enough, in the equation for a free particle (notice: there's no potential here) so in the limit of Cartesian coordinates $\Gamma^m_{lk} \rightarrow 0$ for all components. We've really only found the minimum arclength/pathlength since

$$\delta S = \delta\int ds^2 = \delta\int\left(\frac{ds}{d\lambda}\right)^2 d\lambda^2 \Rightarrow \delta\int \mathcal{T}^{1/2}d\lambda$$

with the $\Gamma$'s representing coordinate induced "accelerations". If we know $g_{ij}$ already, it is straightforward to compute all components of the $\Gamma$ symbols; take, for example, cylindrical coordinates for which:

$$g_{rr} = 1 \,, \quad g_{\varphi\varphi} = r^2 \,, \quad g_{zz} = 1 \quad \therefore$$
$$\mathcal{T} = \tfrac{1}{2}(\dot r^2 + r^2\dot\varphi^2 + \dot z^2)$$

and then:

$$\frac{\partial \mathcal{T}}{\partial r} = \frac{d}{d\lambda}\frac{\partial \mathcal{T}}{\partial \dot r} \Rightarrow \tfrac{1}{2}\frac{\partial}{\partial r} r^2\dot\varphi^2 = \frac{d}{d\lambda}\dot r$$

$$\Rightarrow \quad -r\dot\varphi^2 + \ddot r = 0 \,, \quad \Gamma^r_{\varphi\varphi} = -r \,, \quad \Gamma^r_{ij} = 0 \ \ \forall \text{ other } (i,j)$$

$$\frac{\partial \mathcal{T}}{\partial \varphi} = \frac{d}{d\lambda}\left(\frac{\partial \mathcal{T}}{\partial \dot\varphi}\right) \Rightarrow \frac{d}{d\lambda} r^2\dot\varphi = r^2\ddot\varphi + 2r\dot r\dot\varphi = 0 \,, \quad \Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r}$$

$$\frac{\partial \mathcal{T}}{\partial z} = \frac{d}{d\lambda}\left(\frac{\partial \mathcal{T}}{\partial \dot z}\right) \Rightarrow \ddot z = 0 \,, \quad \Gamma^z_{ij} = 0 \ \ \forall (i,j) \,.$$

If we continue along this line, you can see that $\Gamma^i_{jk} = \Gamma^i_{kj}$ (symmetric in the lower indices) and that if $g_{ij}$ has some direction along which the $\Gamma$'s vanish (independence from the coordinate) the translation is purely "Galilean" — we can set the zero point arbitrarily. This also follows if we introduce $\mathcal{L} = \mathcal{T} - \Phi$, with $\Phi$ being a potential. Then the variational equation yields the Euler–Lagrange equation:

$$\frac{d}{d\lambda}\frac{\partial \mathcal{T}}{\partial \dot x^k} = \frac{\partial}{\partial x^k}(\mathcal{T} - \Phi)$$

since $\Phi$ is independent of $\dot x^l$ for all $l$. The equation:

$$\ddot x^i + \Gamma^i_{jk}\,\dot x^j \dot x^k = 0$$

describes a geodesic and, if we don't know $g_{ij}$ in advance, provides the means for computing it. Well, not quite, since we still have $\ddot x^i$.

Problem For a spherical coordinate system, compute the $\Gamma$'s.