So what really matters in GRT for our problem?

1. There is a problem from the finite age and causality – large scale structure, the physical constants, everything: all are the same everywhere. This has been a problem for a very long time, the smoothness of the initial conditions.
A trick to get around this is to imagine an initial state at the quantum (Planck) scale that is so small it is a <u>single</u> fluctuation. Then the state, if it expands (for some reason) is isotropic and isentropic, with all constants set at what is – in effect – a single point. We'll see this is the basic justification for inflation.

2. There are two scales, one self-bound and the other "global", some global geometry is described by the distribution of matter/energy.

3. In classical physics accelerations are due to gradients in mass/energy density and they're all local. In GRT, there can be a curvature even if no gradients exist. The problem of inertia notwithstanding, we must have a way of describing the interaction of fields with the trajectories we use to measure spacetime.

These are provided by relativity, and one of the most important features is the tool, the curvature, that allows us to link global geometry to local measurement. I won't derive here the rest of the field equations; instead I'll just make an observation. There are two ways to obtain the eqs. for the evolution of the scale factor. Both start with:

$$ds^2 = dt^2 - \frac{R^2}{1-kr^2}dr^2 - R^2 d\varpi^2 \;, \quad d\varpi^2 = r^2 d\theta^2 + r^2 \sin^2\vartheta\, d\varphi^2$$

using $R = R(t)$ only. They assume a very simple equation of state, essentially barotropic, for any mass. <u>How</u> this condition is established is <u>not</u> treated, it's <u>assumed</u> (OK, the point is this word: assumed, <u>ad hoc</u>, requires the initial conditions to be sufficiently smooth and homogeneous that it makes sense to write $P(\rho)$ as a universal function!). We know that the stress tensor is:

$$T^{ij} = \rho u^i u^j + P\delta^{ij}$$

for classical problems and now we'll generalize this to:

$$T^{ij} = \rho u^i u^j + P g^{ij}$$

using the appropriate signature for the metric. We also know that for an empty spacetime (from the Schwarzschild solution), $R_{ij} = 0$ which would give us – for a case <u>not</u> the same as a point mass – the solution $g_{ij} = \eta_{ij}$, a Minkowski metric. In other words, without a mass distribution, the solution is trivial and uninteresting. But we have $(P,\rho) \neq 0$. Thus

$$G_{ij} \equiv R_{ij} - \tfrac{1}{2}Rg_{ij} = -\kappa T_{ij}$$

is the usual form for the field equations. Because we have a diagonal metric, one for which only time derivatives will ultimately matter, we can use this to write a set of equations for g_{ij}. We can also go in the other direction:

$$R_{ij} \approx T_{ij} - \tfrac{1}{2}g_{ij}T \,,$$

which was the path used by Einstein for the field equations. These two are completely equivalent. Then:

$$R_{ij} = -\Gamma^{k}_{ij,k} + \Gamma^{k}_{ik,j} - \Gamma^{m}_{ij}\Gamma^{k}_{mk} + \Gamma^{l}_{ik}\Gamma^{k}_{lj}$$

Begin with the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\left[\dot{t}^2 - R^2(t)F(r)\dot{r}^2 - R^2(t)r^2\dot{\omega}^2 \right]$$

where $\dot{\omega}^2 = \dot{\vartheta}^2 + \sin^2\vartheta\,\dot{\varphi}^2$. We'll see what $F(r)$ is shortly (well, you already know this but for now let's leave it general). The "$\cdot$" will have two different meanings now, to denote $\frac{d}{d\lambda}$ for some affine parameter and $\frac{d}{dt}$ for $R(t)$ using:

$$\frac{dR}{d\lambda} = \frac{dR}{dt}\dot{t}$$ (sorry for the confusion but I get lazy sometimes)

From the variational equations:

$$\frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial\dot{x}^i} = \frac{\partial\mathcal{L}}{\partial x^i} \qquad x^i = (t, r, \vartheta, \varphi)$$

we have the geodesics and, consequently, the connections Γ^i_{jk}. For instance,

$$\frac{d}{d\lambda}\dot{t} = \frac{\partial}{\partial t}\left(-\frac{1}{2}R^2(t)F\dot{r}^2 - \frac{R^2}{2}r^2\dot{\omega}^2\right)$$

from which we have:

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{\dot{R}}{R}, \quad \Gamma^0_{11} = R\dot{R}F, \quad \Gamma^0_{22} = \frac{\dot{R}}{R}r^2, \quad \Gamma^0_{33} = \frac{\dot{R}}{R}r^2\sin^2\vartheta$$

and:

$$\Gamma^1_{01} = \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{R}}{R}$$

$$\Gamma^1_{11} = \frac{1}{2}\frac{F'}{F}, \quad \Gamma^1_{22} = -\frac{r}{F}, \quad \Gamma^1_{13} = -\frac{r\sin^2\vartheta}{F} = \Gamma^1_{31}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$$

$$\Gamma^2_{33} = -\sin\vartheta\cos\vartheta, \quad \Gamma^3_{32} = \Gamma^3_{23} = \cot\vartheta$$

with, as you'll be able to show for yourselves, all other symbols vanishing. Then:

$$R_{ij} = \Gamma^k_{ik,j} - \Gamma^k_{ij,k} + \Gamma^m_{in}\Gamma^n_{jm} - \Gamma^m_{ij}\Gamma^k_{mk},$$

the usual summation convention holding here. Only diagonal terms hold for R_{ij}, all off diagonal terms vanish, and the trace is the curvature scalar with all sums being taken over g^{ii}, itself also diagonal. For the simplest:

$$R_{00} = \Gamma^k_{0k,0} + (\Gamma^k_{0k})^2 = 3\left(\frac{\dot{R}}{R}\right)_{,0} + 3\left(\frac{\dot{R}}{R}\right)^2 = 3\frac{\ddot{R}}{R}$$

and for the next,

$$R_{11} = \Gamma^2_{12,1} + \Gamma^3_{13,1} - \Gamma^0_{11,0} - \Gamma^1_{11}(\Gamma^2_{12} + \Gamma^3_{13}) + (\Gamma^0_{10})^2 + (\Gamma^2_{12})^2 + (\Gamma^3_{13})^2$$

$$= \left(\frac{R\ddot{R} + 2\dot{R}^2 + 2k}{R^2}\right)\frac{1}{1-kr^2}$$

Now for the stress tensor, we have to choose an observer at rest with respect to the universe so

$$T_{ij} = \rho u_i u_j - P g_{ij}$$ (classically $T_{ij} = \rho v_i v_j - P\delta_{ij}$, see Astrofisica II)

(taking here $u_0 = c$ and the

Isometries (a short note)

The metric of a surface may depend on $\{x^i\}$ (not for η_{ij} but that's a special case in cartesian coordinates), but if g_{ij} is invariant, then we can find a set of coordinates that leave it invariant. Take

$$g_{ij}(x) = g_{ij}(\bar{x}+\xi) \quad \text{for some} \quad x \to \bar{x}+\xi$$

where ξ is a displacement (in any number of dimensions, so ξ isn't necessarily a vector).
First :

$$\bar{g}(\bar{x}) \equiv \bar{g}_{ij} = g_{kl}\frac{\partial x^k \partial x^l}{\partial \bar{x}^i \partial \bar{x}^j}$$

and we note that $\frac{\partial x^i}{\partial x^m} = \delta^i_m$; then

$$\bar{g}_{ij} = g_{im}\xi^m_{,j} + g_{jm}\xi^m_{,i}$$

and $g_{ij}(\bar{x}+\xi) = \frac{\partial g_{ij}}{\partial x^m}\xi^m + g_{ij}(\bar{x})$; so if $g = \bar{g}$, the coordinates that leave g invariant are given by ξ^m. These are simplified as:

$$(g_{im}\xi^m)_{,j} = g_{im,j}\xi^m + g_{im}\xi^m_{,j} \qquad \text{recall: } g_{lm}\xi^m = \xi_l$$

$$\to (g_{im}\xi^m)_{,j} + (g_{jm}\xi^m)_{,i} + 2(g_{ij,m} - g_{im,j} - g_{jm,i})\xi^m$$

$$\to \xi_{i;j} + \xi_{j;i} = 0,$$

where we've used the definition of the Christoffel symbol and the property of the metric to raise/lower indices. The resulting Killing vectors (ξ_i) are the directions we've sought (after Killing, who investigated Lie groups in the '800). For a symmetric g this is comparatively easy to solve since:

$$g_{ij} = g_{ji} \to g_{im,j}\xi^m = g_{mm}\xi^m_{,j}\delta_{mn}$$

For instance, if we take a spherical surface, then $g_{22} = r^2$ and $g_{33} = r^2\sin^2\vartheta$. Thus motion around φ must be an invariant direction and g_{11} can't depend on it.

Now we'll use this. Consider a stationary metric (no t dependence) and we'll use the form of the Killing equations:

$$g_{ij,m}\xi^m + g_{im}\xi^m_{,j} + g_{jm}\xi^m_{,i} = 0$$

for a spherical metric with $g_{22} = r^2$. Then

$$g_{11,m}\xi^m + g_{1m}\xi^m_{,1} + g_{m1}\xi^m_{,1}$$

$$= g_{11,m}\xi^m + 2g_{11}\xi^1_{,1} = 0$$

$$g_{22,m}\xi^m + g_{2m}\xi^m_{,2} + g_{2m}\xi^m_{,2} = g_{22,m}\xi^m + 2g_{22}\xi^2_{,2} = 0$$

$$g_{11}\xi^1_{,2} + g_{22}\xi^2_{,1} = 0 \quad (g_{12} = g_{21} = 0)$$

From the first equation, we get:

$$\xi^1\frac{\partial g_{11}}{\partial r} = -2g_{11}\frac{\partial \xi^1}{\partial r} \to \xi^1 = g_{11}^{-1/2}\Theta(\vartheta)$$

since the constant is undetermined. Then from the third equation:

$$g_{11}^{1/2}\Theta' + r^2\xi^2_{,1} = 0$$

and therefore from the second equation and third equation

$$-\left(\frac{\xi'}{r}\right)_{,1} = \frac{g_{11}^{1/2}}{r^2}\,\Theta'' = -\left(\frac{\xi'}{r}\right)_{,1}\Theta \qquad \left(' \equiv \frac{\partial}{\partial\vartheta}\right)$$

Calling $u \equiv \frac{g_{11}^{1/2}}{r}$ we have:

$$u_{,1} = -\frac{u}{r}\frac{\Theta''}{\Theta} \rightarrow \frac{\partial \ln u}{\partial \ln r} = -k$$

(independent of r, constant for that purpose.)

so that

$$g_{11}^{1/2} = (1-kr^2)^{-1} \quad \text{(well, actually, this is scaled)}$$

and consequently:

$$d\tilde{s}^2 = \frac{dr^2}{1-kr^2} + r^2 d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2 .$$

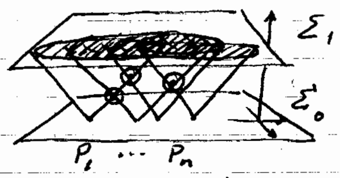
Thus for $k=0$ (a free choice) we obtain a conventional spherical metric, while for $k>0$ or $k<0$ we have alternate forms for the metric distances.

⇒ As a note, we haven't really needed the covariant form of the Killing equation since we've used the metric explicitly. Once we have g, we can always compute the appropriate coefficients (Γ) and raise and lower indices as needed.

Now back to our main line of development

In classical theory, in which we don't need to worry about spacetime or causality, it's simple enough to assert that we have a global potential due to some mass distribution in an infinite universe and solve the field equation (Poisson). Initial conditions are not a problem. In a static universe, this also isn't a problem. But the redshift, whatever we take its cause to be, means time (for a comoving observer) matters and we have a problem: if two points are separated now by a distance on the sky, they are within our event horizon. But at some time in the past they weren't.

In this discussion, I'm not intending to derive all of GRT or even any large subsection. That's available in Ch. I of "Tapestry". But before we go into the Friedmann equations and their various solutions, a word is needed about the consequences of certain features of the theory. In classical physics, where c is not a critical limiting factor (the postulate of simultaneity), any two points in space are connected by a potential that gets established everywhere "instantaneously". Thus, Newton had no problem with an isotropic, homogeneous universe and – since the global structure was "fixed" forever, there was enough time for all parts to homogenize. In GRT, though, this all changes. The theory must be causal – hence, a spacetime – and it must evolve with time, itself defined by the observer. If at some time t we see, on our surface Σ_1 that any set of points $\{P\}$ are uniform, at some time $t_0 \ll t$ these points – in an evolving spacetime – must have been separated by superluminal distances. The problem of an initial condition is, consequently, much harder in GRT cosmology since we can easily go "in reverse" (from $t \to t_0$) but it's harder to go from $t_0 \to t$ for a set of points $\{P\}$. In effect, the problem can be stated this way: you know that the present day universe is completely homogenized, that the covariance principle holds and we can really form a continuous atlas – a manifold – characterized by a continuous metric. So how is this possible?



An auxillary problem is far simpler: is there a way to specify the global geometry? The FRW equations (Friedmann - Robertson - Walker) evolution eqs, allow this with a caveat. The assumption will be that

$$g_{ij}^{(e)}(t) \to R(t)\, g_{ij}$$

where $R(t)$ is a scalar function that merely scales $g \to g^{(e)}$. In other words, if we have a metric g imposed on Σ, then at any time $t > t_0$ we have the same surface only with all metric distances collectively scaled by a simple multiplicative factor that depends only on the proper time (the affine parameter). The "equations of motion", the second order system that governs $R(t)$ leaves all components of g in the hypersurface relatively unaltered, you can think of this as we did in lecture as "geometric figures on Σ_0 map to Σ_1 with no changes in angles".

This last property is most naturally expressed using the invariance of the metric on Σ. You see this by taking, in a schematic form:

$$g_{ij}(\bar{x}+\xi) \simeq g(\bar{x}) + \frac{\partial g}{\partial x}\xi$$

but now noting that

$$\bar{g}_{ij} = g_{ij}(\bar{x}) = g_{ij}\frac{\partial x^i}{\partial \bar{x}^l}\frac{\partial x^j}{\partial \bar{x}^m}$$

The simplest argument for the redshift comes from the metric: when $ds^2=0$, we're following light here, the metric gives:

$$\int \frac{dr}{(1-kr^2)^{1/2}} = \int \frac{dt}{R}.$$

This means if we take two points at a distance δr apart, this scales with the expansion factor $R(t)$, this is the proper distance for the object. Thus consider two pulses, one emitted at t_e and the next at $t_e+\delta t_e$. The first will be received at t_r, the second some time later at $t_r+\delta t_r$. Then:

$$\int_{t_e}^{t_r} \frac{dt}{R} = \int_{t_e+\delta t_e}^{t_r+\delta t_r} \frac{dt}{R}.$$

Now notice that by continuity, we have $\int_{t_e}^{t_r} = \int_{t_e}^{t_e+\delta t_e} + \int_{t_e+\delta t_e}^{t_r}$, and similarly for $(t_e+\delta t_e,\ t_r+\delta t_r)$ so that

$$\int_{t_e}^{t_e+\delta t_e} + \int_{t_e+\delta t_e}^{t_r} = \int_{t_e+\delta t_e}^{t_r} + \int_{t_r}^{t_r+\delta t_r},$$

and thus:

$$\int_{t_e}^{t_e+\delta t_e} = \int_{t_r}^{t_r+\delta t_r}$$

If we make δt so short it corresponds to ~~as~~ ν^{-1}, the frequency, then:

$$\int_{t_e}^{t_e+\delta t_e}\frac{dt}{R} \sim \frac{\delta t_e}{R(t_e)}\ ,\quad \int_{t_r}^{t_r+\delta t_r}\frac{dt}{R} \sim \frac{\delta t_r}{R(t_r)}$$

and since $\delta t \doteq \delta x^0$ ($c=1$ for convenience) and these two observers are relatively at rest:

$$\frac{\delta t_e}{R(t_e)} = \frac{\delta t_r}{R(t_r)} \rightarrow \frac{\lambda_e}{R(t_e)} = \frac{\lambda_r}{R(t_r)} \rightarrow \frac{\lambda}{R} = \frac{\lambda_0}{R_0} \therefore$$

$$\frac{\lambda}{\lambda_0} = \left(\frac{R_e}{R_0}\right)^{-1} \rightarrow (1+z) \equiv \frac{R_0}{R}$$

from the observer's definition of z. For $R^{-1}(t)$, expand this as a power series using the definition

$$q_0 \equiv -\frac{\ddot{R}_0 R_0}{\dot{R}^2} = -\frac{\ddot{R}_0}{R_0}H_0^{2}\ ,\quad H_0 \equiv \frac{\dot{R}_0}{R_0}$$

so that

$$z \simeq H_0 t + (2q_0+1)H_0^2 t^2 \quad (\Delta t \rightarrow t \text{ for } t_0=0)$$

which gives t at every z, the scale factor having been absorbed into (q_0, H_0). In other words, independent of the details of the evolution of $R(t)$, if (H_0, q_0) are constants we have a way to go from the measured redshift to a time, called the "lookback time". This ~~holds~~ follows simply for observers at rest - those for which $d\omega^2=0$ so the "sky" each observes doesn't change, the angular separations remain the same while the linear (metric) distance increases.

NB To recapitulate the discussion from today about structure formation, recall the point that the rate of dilution is H_0 so H_0^{-1} is the timescale. For the analog of the Jeans' instability, but now in an expanding background,

$$\frac{t_{sf}}{t_{expansion}} \sim \frac{H_0}{(G\rho_0)^{1/2}} \rightarrow \rho_c \sim \frac{H_0^2}{G}$$

is an estimate of the critical density, independent of the temporal behavior of

$R(t)$. Now if we have $\rho \ll \rho_c$ and imagine density fluctuations (uncorrelated) due to a gaussian spectrum, only the most extreme fluctuations $\delta\rho(\sigma)$, with σ being $\langle \delta\rho^2 \rangle^{1/2}$, will reach or exceed ρ_c. Thus, we expect a bias in this case for structure. It should correspond to only the most extreme density perturbations. If, on the other hand, $\rho \sim \rho_c$, then only a small $\delta\rho$ is needed to produce gravitational collapse. Obviously, since

$$\rho \sim R^{-3} \quad \text{non-relativistic (volume} = d^3x)$$
$$\sim R^{-4} \quad \text{relativistic (volume} = d^3x\,dt)$$

then the era of structure formation is relatively short and depends on how rapidly $R(t)$ is changing (the value of (H_0, q_0)). The rate of accretion of matter by a structure also depends on the mass:

$$\dot{m} \sim \frac{G^2 m^2}{\sigma_v^3}\rho \qquad \sigma_v = \text{velocity dispersion}$$

and if this is about the same as the the rate of expansion:

$$\left.\frac{\dot{m}}{m}\right)_c \approx H_0 \sim \frac{m}{\sigma_v^3}\rho$$

then we have a relation between σ_v and the mass of the structures, a possible non-virial explanation of the L-σ_v calibrators (the Tully-Fisher and Faber-Jackson relations). At any rate, the critical density is obviously just that required to "turn an expansion around, locally", or as I've repeatedly said, to have a neighborhood (bound structure). Obviously, if the material collapsing into a denser structure fragments (and it must) without dissipation (and this may be true) then the smallest scale structures - galaxies - must have much higher densities than ρ_c. Consequently, by simple extension, any bodies they internally form will be very tightly bound (stars).

For further application, we know for the photons that

$$\rho \sim T^4 \rightarrow \rho \sim R^{-4} \;\&\; T \sim R^{-1};$$

this implies $T \sim T_0(1+z)$ so if we have an effective decoupling when $\kappa\rho R \gtrsim 1$ then if $\kappa = \sigma_T$, this is when $\rho_0(1+z)^2 \geq 1$ and for $T_{CBR} \sim$ a few $\times 10^3$ K we have $z \sim 10^3$ at decoupling of matter and radiation. Alternately, when:

$$\rho_{r0}(1+z_*)^4 \sim \rho_{m0}(1+z_*)^3 \rightarrow \frac{\rho_{m0}}{\rho_{r0}} \sim (1+z_*)$$

Finally, if the epoch is always matter-dominated, at $z \gtrsim 10^6$ we have an epoch when ρ_m is high as is the temperature, hence the nucleosynthetic era.

$$\dot r^2 R^2 F \to d_\lambda(\dot r R^2 F) = 2\dot r R\dot R F\dot t + \dot r^2 F' R^2 + R^2 F\ddot r$$
$$= R^2F\left[2\frac{\dot R}{R}\dot r\dot t + \frac{F'}{F}\dot r^2 + \ddot r\right] = \frac12 R^2 F'\dot r^2 + r\dot\omega^2 R^2$$
$$0 = \frac{2\dot R}{R}\dot r\dot t + \frac12\frac{F'}{F}\dot r^2 + \ddot r - \frac{1}{F}\dot\omega^2$$

$$d_\lambda(R^2r^2\dot\vartheta) = R^2r^2\ddot\vartheta + 2R\dot R\dot\vartheta\dot t r^2 + 2rR^2\dot r\dot\vartheta - R^2r^2\cos\vartheta\sin\vartheta\dot\varphi^2 = 0$$
$$\ddot\vartheta + 2\frac{\dot R}{R}\dot\vartheta\dot t + \frac{2}{r}\dot\vartheta\dot r - \cos\vartheta\sin\vartheta\dot\varphi^2 = 0$$

$$d_\lambda(R^2r^2\sin^2\vartheta\dot\varphi) = 0 \to R^2r^2\sin^2\vartheta\ddot\varphi + 2R\dot Rr^2\sin^2\vartheta\dot\varphi\dot t + 2R^2\sin\vartheta\cos\vartheta\dot\vartheta\dot\varphi r^2 + 2rR^2\sin^2\vartheta\dot r\dot\varphi$$
$$= r^2R^2\sin^2\vartheta\left(\ddot\varphi + 2\frac{\dot R}{R}\dot\vartheta\dot t + 2\cot\vartheta\dot\vartheta\dot\varphi + \frac{2}{r}\dot r\dot\varphi\right) = 0$$

$\Gamma^3_{03} = \frac{\dot R}{R}$ $\quad \Gamma^3_{13} = \frac{1}{r}$ $\quad \Gamma^2_{23} = \cot\vartheta$

$\Gamma^2_{02} = \frac{\dot R}{R}$ $\quad \Gamma^2_{12} = \frac{1}{r}$ $\quad \Gamma^2_{33} = -\cos\vartheta\sin\vartheta$

$\Gamma^1_{01} = \frac{\dot R}{R}$ $\quad \Gamma^1_{11} = \frac12\frac{F'}{F}$ $\quad \Gamma^1_{22} = \frac{1}{r}$ $\quad \Gamma^1_{33} = \frac{\sin^2\vartheta}{r}$

$$F = (1-kr^2)^{-1} \to \frac12\frac{F'}{F} = \frac{kr}{1-kr^2}$$

$\Gamma^k_{ii,k}$ $\quad \Gamma^1_{11,1},\ \Gamma^2_{33,2},\ \Gamma^1_{33,1},\ \Gamma^1_{33,1},\ \Gamma^0_{22,0},\ \Gamma^0_{33,0}$

$\Gamma^k_{ik,i}$ $\quad \Gamma^0_{10,1},\ \Gamma^1_{01,0},\ \Gamma^2_{02,0},\ \Gamma^3_{03,0},\ \Gamma^2_{12,1},\ \Gamma^3_{13,1},\ \Gamma^0_{11,0}$

$\cancel{\Gamma^0_{20,2}}$ $\quad \cancel{\Gamma^1_{21,2}}$ $\quad \cancel{\Gamma^2_{22,2}}$ $\quad \Gamma^3_{23,2}$

$\cancel{\Gamma^0_{30,3}}$ $\quad \cancel{\Gamma^1_{31,3}}$ $\quad \cancel{\Gamma^2_{32,3}}$ $\quad \cancel{\Gamma^3_{33,3}}$

$$\ddot t = -\cancel{2}R\dot RF\dot r^2 - \cancel{2}R\dot Rr^2\dot\omega^2 \to \ddot t + \cancel{2}R\dot RF\dot r\dot t + \cancel{2}R\dot Rr^2\dot\vartheta^2 + \cancel{2}R\dot Rr^2\sin^2\vartheta\dot\varphi^2$$

$\Gamma^0_{11} = R\dot RF$ $\quad \Gamma^0_{22} = R\dot Rr^2$ $\quad \Gamma^0_{33} = R\dot Rr^2\sin^2\vartheta$

$\cancel{\Gamma^0_{10,1} = \frac12 R\dot RF'}$ $\quad \Gamma^0_{22,0} = \Gamma^0_{33,0}/\sin^2\vartheta$ $\quad \Gamma^0_{11,1} = R\dot RF'$

$\Gamma^k_{ii,k} - \Gamma^k_{ik,i} \Rightarrow$ $\quad \Gamma^j_{00} = 0,\ \Gamma^k_{11}, \Gamma^k_{22}, \Gamma^k_{33} \neq 0$

$\Gamma^k_{0k,0} = 3\Gamma^1_{01,0} = 3\frac{\ddot R}{R} - 3\left(\frac{\dot R}{R}\right)^2$ $\therefore$ $\quad R_{00} = 3\frac{\ddot R}{R}$

$(\Gamma^k_{0k})^2 = 3\left(\frac{\dot R}{R}\right)^2$

$\Gamma^j_{11} = \frac12\frac{F'}{F}$ $\quad (\Gamma^k_{1k})^2 = (\Gamma^0_{10})^2 + (\Gamma^1_{11})^2 + (\Gamma^2_{12})^2 + (\Gamma^3_{13})^2$

$$= \left(\frac{\dot R}{R}\right)^2 + \frac12\left(\frac{F'}{F}\right)^2 + \frac{2}{r^2}$$

$\Gamma^k_{11,k} = \Gamma^1_{11,1} = \frac12\left(\frac{F'}{F}\right)' + \Gamma^0_{11,0}$ $\qquad F' = \frac{2kr}{(1-kr^2)^2}$

$\Gamma^k_{1k,1} = \Gamma^1_{11,1} + \cancel{\Gamma^0_{10,1}} + \Gamma^2_{12,1} + \Gamma^3_{13,1}$

$= \frac12\left(\frac{F'}{F}\right)' + \cancel{R\dot RF'} - \frac{2}{r^2}$ $\qquad \Gamma^0_{11,0} = (R\dot R)^{\cdot}$

$$\Gamma^k_{ik,j} - \Gamma^k_{ij,k} + \Gamma^k_{il}\Gamma^l_{kj} - \Gamma^k_{ij}\Gamma^l_{kl} = R_{ij}$$

$$\Rightarrow \Gamma^k_{ik,i} - \Gamma^k_{ii,k} + \Gamma^k_{il}\Gamma^l_{ki} - \Gamma^k_{ii}\Gamma^l_{kl}$$

$$\Gamma^k_{1k,1} - \Gamma^k_{11,k} + \Gamma^k_{1l}\Gamma^l_{k1} - \Gamma^k_{11}\Gamma^l_{kl}$$

$$\cancel{\Gamma^0_{10,1}} + \Gamma^1_{11,1} + \Gamma^2_{12,1} + \Gamma^3_{13,1} - \Gamma^0_{11,0} - \Gamma^1_{11,1} - \Gamma^2_{11,2} - \Gamma^3_{11,3}$$

$$\cancel{\frac{1}{2}\left(\frac{F'}{F}\right)' + \frac{1}{2}R\dot{R}F' - \frac{2}{r^2} - \frac{1}{2}\left(\frac{F'}{F}\right)'}$$

$$\Gamma^k_{10}\Gamma^0_{k1} + \Gamma^k_{11}\Gamma^1_{k1} + \Gamma^k_{12}\Gamma^2_{k1} + \Gamma^k_{13}\Gamma^3_{k1}$$

$$= \cancel{\Gamma^0_{10}\Gamma^0_{01}} + \Gamma^1_{10}\Gamma^0_{11} + \cancel{\Gamma^2_{10}\Gamma^0_{21}} + \cancel{\Gamma^3_{10}\Gamma^0_{31}}$$

$$\Gamma^0_{11}\Gamma^1_{01} + \cancel{\Gamma^1_{11}\Gamma^2_{11}} + \cancel{\Gamma^2_{11}\Gamma^1_{21}} + \cancel{\Gamma^3_{11}\Gamma^1_{31}}$$

$$\cancel{\Gamma^0_{12}\Gamma^2_{01}} + \cancel{\Gamma^1_{12}\Gamma^2_{11}} + \Gamma^2_{12}\Gamma^2_{21} + \cancel{\Gamma^3_{12}\Gamma^2_{31}}$$

$$\cancel{\Gamma^0_{13}\Gamma^3_{01}} + \cancel{\Gamma^1_{13}\Gamma^3_{11}} + \cancel{\Gamma^2_{13}\Gamma^3_{21}} + \Gamma^3_{13}\Gamma^3_{31}$$

$$= \cancel{(\Gamma^0_{10})^2} \quad \Gamma^1_{10}\Gamma^0_{11} + \Gamma^0_{11}\Gamma^1_{01} + \Gamma^2_{12}\Gamma^2_{12} + \Gamma^3_{13}\Gamma^3_{13}$$

$$\Gamma^0_{11}\Gamma^l_{0l} + \Gamma^1_{11}\Gamma^l_{1l} + \cancel{\Gamma^2_{11}\Gamma^l_{2l}} + \cancel{\Gamma^3_{11}\Gamma^l_{3l}}$$

$$= \Gamma^0_{11}\Gamma^1_{01} + \Gamma^0_{11}\Gamma^2_{02} + \Gamma^0_{11}\Gamma^3_{03} + \Gamma^1_{11}\Gamma^1_{11} + \Gamma^0_{11}\Gamma^0_{10} + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^1_{11}\Gamma^3_{13}$$

$$\Gamma^k_{10}\Gamma^0_{k1} - \Gamma^k_{11}\Gamma^l_{kl} = \cancel{\Gamma^1_{10}\Gamma^0_{11}} + \Gamma^0_{11}\Gamma^1_{10} + \Gamma^2_{12}\Gamma^2_{12} + \Gamma^3_{13}\Gamma^3_{13}$$

$$- \cancel{\Gamma^0_{11}\Gamma^1_{01}} - \Gamma^0_{11}\Gamma^2_{02} - \Gamma^0_{11}\Gamma^3_{03} - \Gamma^1_{11}\Gamma^1_{11}$$

$$- \cancel{\Gamma^0_{11}\Gamma^0_{10}} - \Gamma^1_{11}\Gamma^2_{12} - \Gamma^1_{11}\Gamma^3_{13}$$

$$\Gamma^1_{11,1} + 2\Gamma^2_{12,1} - \Gamma^0_{11,0} + \Gamma^0_{11}\Gamma^1_{10} + 2\Gamma^2_{12}\Gamma^2_{12} - \Gamma^0_{11}\Gamma^2_{02} - \Gamma^0_{11}\Gamma^3_{03} - \Gamma^1_{11}\Gamma^1_{11}$$
$$-2\Gamma^1_{11}\Gamma^2_{12}$$

$$\frac{1}{2}\left(\frac{F'}{F}\right)' - \frac{2}{r^2} - (R\dot{R})_{,0}F + R\dot{R}F\frac{\dot{R}}{R} - \frac{1}{2}\frac{F'}{F}\frac{\dot{R}}{R}\frac{1}{r} + \frac{2}{r^2} - 2R\dot{R}F\frac{\dot{R}}{R} - \frac{1}{4}\left(\frac{F'}{F}\right)^2$$
$$- \frac{F'}{F}\frac{1}{r}$$

$$= \frac{1}{2}\left(\frac{F'}{F}\right)' - (R\dot{R})_{,0}F - R\dot{R}\frac{\dot{R}}{R}F - \frac{1}{2}\frac{F'}{F}\frac{\dot{R}}{R} - \frac{1}{4}\left(\frac{F'}{F}\right)^2 - \frac{1}{r}\frac{F'}{F}$$

$$= \left(\frac{kr}{1-kr^2}\right)' - \frac{(R\dot{R})_{,0}}{(1-kr^2)} - \frac{\dot{R}^2}{(1-kr^2)} - \frac{kr}{1-kr^2}\frac{\dot{R}}{R} - \frac{k^2r^2}{(1-kr^2)^2} - \frac{2k}{(1-kr^2)}$$

$$= \frac{k}{1-kr^2} + \frac{k^2r^2}{(1-kr^2)^2} - \frac{(R\dot{R})_{,0}}{1-kr^2} - \frac{kr}{1-kr^2}\frac{\dot{R}}{R} - \frac{\dot{R}^2}{1-kr^2} - \frac{2k}{1-kr^2}$$

$$= -\frac{2k}{1-kr^2} - \frac{(R\dot{R})_{,0}}{1-kr^2} + \ldots = -\left(\frac{R\ddot{R} + 2\dot{R}^2 + 2k}{R^2}\right)\frac{1}{1-kr^2}$$
$$- \frac{\dot{R}^2}{1-kr^2}$$

$$R_{22} = \Gamma^{k}_{2k,2} - \Gamma^{k}_{22,k} + \Gamma^{k}_{2l}\Gamma^{l}_{k2} - \Gamma^{k}_{22}\Gamma^{l}_{kl}$$

$$\cancel{R_{33} = 2\Gamma^{k}_{3k}} \quad -\Gamma^{k}_{33,k} + \Gamma^{k}_{3l}\Gamma^{l}_{3k} - \Gamma^{k}_{33}\Gamma^{l}_{kl} = R_{33}$$

$$-\Gamma^{0}_{22,0} - \Gamma^{1}_{22,1} + \left(\Gamma^{k}_{20}\Gamma^{0}_{k2} + \Gamma^{k}_{21}\Gamma^{1}_{k2} + \Gamma^{k}_{22}\Gamma^{2}_{k2} + \Gamma^{k}_{23}\Gamma^{3}_{k2}\right)$$
$$- \left(\Gamma^{0}_{22}\Gamma^{l}_{0l} + \Gamma^{1}_{22}\Gamma^{l}_{1l} + \Gamma^{2}_{22}\Gamma^{l}_{2l} + \Gamma^{3}_{22}\Gamma^{l}_{3l}\right)$$

$$\cancel{-2\Gamma^{0}_{22,0}}$$

$$= -(R\dot{R})_{,0}\, r^2 + \frac{1}{r^2} + \left(\cancel{\Gamma^{0}_{20}\Gamma^{0}_{02}} + \Gamma^{2}_{21}\Gamma^{1}_{22} + \Gamma^{0}_{22}\Gamma^{2}_{02}\right) \cancel{\Gamma^{2}_{23}\Gamma^{3}_{22}})$$
$$- \left(2\Gamma^{0}_{22}\Gamma^{2}_{02} + \Gamma^{1}_{22}\Gamma^{1}_{11} + 2\Gamma^{1}_{22}\Gamma^{2}_{12}\right) \cancel{F}$$

$$= -(R\dot{R})_{,0}\, r^2 + \frac{1}{r^2} + \left(\frac{1}{r^2} + 2R\dot{R}r^2\frac{\dot{R}}{R}\right)$$
$$- \left(2\frac{1}{r}\frac{R'}{R} + \frac{1}{r}\cdot\frac{1}{2}\frac{F'}{F} + 2\frac{1}{r^2}\right)$$

$$= -(R\dot{R})_{,0}\, r^2 + \cancel{\frac{1}{r^2}} + \cancel{\frac{1}{r^2}} + 2\dot{R}^2 r^2 - 2\frac{1}{r}\frac{R'}{R} - \cancel{\frac{1}{r}}\cdot\frac{k\cancel{r}}{1-kr^2} - \cancel{\frac{2}{r^2}}$$

$$= -R\ddot{R}r^2 - \dot{R}^2 r^2 + 2\dot{R}^2 r^2 - 2\frac{R'}{R}\frac{1}{r} - \frac{k}{1-kr^2}$$

$$= -R\ddot{R}r^2 + \dot{R}^2 r^2 - kr^2$$

$$R_{22} = R_{11}\, r^2(1-kr^2)$$
$$R_{33} = R_{11}\, r^2(1-kr^2)\sin^2\vartheta$$

$$R_{00} - (1-kr^2)R_{11} - r^{-2}R_{22} - r^{-2}(\sin^2\vartheta)^{-1}R_{33} = R$$

To repeat a point from today's session, take the conservative form of the fluid equations:

$$\partial_t \rho + \partial_k J^k = 0$$
$$\partial_t \rho v^k + \partial_k T^{ik} = 0$$

without being respectful of the metric, for the moment. These "smell" like equations for a spacetime, having the form $(\partial_0 J^0 + \partial_k J^k)$ and $(\partial_0 T^{0i} + \partial_k T^{ik})$, therefore suggesting we can write the current (J) and stress tensor (T) in some 4-space form. You might infer this from the classical stress tensor:

$$T_{ij} = \rho v_i v_j + P\delta_{ij}$$

and imagine that if we have v^0 = constant, since $T_{ij} = T_{ji}$, we can write this in a covariant form:

$$T^{ik}{}_{;k} = 0$$

which now holds in any coordinate system. Note that this guess applies also to the current,

$$J^k{}_{;k} = 0$$

so it too is divergenceless. We can also guess that since

$$(R_{ij} - \tfrac{1}{2} g_{ij} R)_{;j} = 0$$

and from the definition of the metric that

$$g_{ij;j} = 0 \quad (g^{ij}{}_{;j} = 0)$$

that

$$R_{ij} - \tfrac{1}{2} g_{ij} R = \kappa T_{ij}$$

since then κ is a Lagrange multiplier. Since R_{ij} depends on the second derivatives of g_{ij}, which itself is dimensionless (defined by the scalar product of unit vectors) then the combinations of c and G, the only dimensioned physical constants that enter the field equations for gravity, must cancel this dependence of T on mass and time. This means $\kappa \sim \frac{G}{c^4}$ and from the Poisson equation we have a factor of 8π:

$$\kappa = \frac{8\pi G}{c^4}, \text{ with } v^0 = c. \rightarrow \underset{(c \to 1)}{\kappa = 8\pi G},$$

and we also have the option of normalizing to a length scale (through the gravitational event horizon, $G c^{-2}$ for a fixed mass - also called geometrical units)

But we can do more. For instance, if we have an isotropic space without shear, then: $T_{ij} = T_{ji} \rightarrow T_{ii}$ (no sum intended); only diagonal components enter. Then, since we know that g_{ij} is also diagonal, so must R_{ij} be. We thus reduce the problem to only 4 components of the Ricci tensor. In fact, there are really only 2 since the space and time separation $(3+1)$ is affected for an isotropic spacetime, so R_{00} and R_{11} suffice. Both may depend on k, the radius of curvature (since $(1-kr^2)$ must be dimensionless - i.e. $F(r)$ must be dimensionless in general - k has the dimensions of the curvature (R^{-2} for some R, not any of the forms we've used so far.)

From the starting point of the Friedmann - Robertson - Walker (and, really, also Tolman) equations, we have:

$$R_{ij} - \tfrac{1}{2} g_{ij} R = 8\pi G\, T_{ij}$$

(this was inverted in our discussions where when writing the equations I used instead

$$R_{ij} = 8\pi G \left(T_{ij} - \tfrac{1}{2} g_{ij} T \right)$$

with T ~~bldg~~ representing the trace $(\rho - 3P)$ of T_{ij}). Then from

$$R = -6\left(\frac{\ddot R}{R} + \left(\frac{\dot R}{R}\right)^2 + \frac{k}{R^2} \right)$$

and, as you found:

$$R_{ij} = -\left(\frac{\ddot R}{R} + 2\left(\frac{\dot R}{R}\right)^2 + \frac{2k}{R^2} \right) g_{ij} \qquad (i,j \in [1,2,3])$$

$$R_{00} = -3\,\frac{\ddot R}{R}$$

then

$$\left(\frac{\dot R}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho$$

$$2\frac{\ddot R}{R} + \left(\frac{\dot R}{R}\right)^2 + \frac{k}{R^2} = -8\pi G P$$

and combining we have:

$$\frac{\ddot R}{R} = -\frac{4\pi G}{3}(3P + \rho).$$

Finally, we add the equation for the energy:

$$\frac{d}{dt}(\rho R^3) + P\frac{d}{dt}R^3 = 0.$$

Also: $T dS = dE + PdV = d(\rho V) + PdV$; since $T\frac{dP}{dT} = P + \rho$ $\left(\frac{\partial^2 S}{\partial V \partial T} = \frac{\partial^2 S}{\partial T \partial V} \right)$ we have $d\left(\frac{1}{T}(\rho + P) R^3 \right) = 0$

Now we have, except for the term Λg_{ij}, the complete FRW equations. Once we have chosen an equation of state, one that relates P and ρ, we have a completely closed system. Always, please, keep in mind my "discource" from Monday — these equations <u>resemble</u> the Newtonian equations of motion, but only if you consider finite volumes in a <u>very</u> particular universe (a sphere) and then only at the risk of misidentifying R(t) with a "trajectory". It's not, however, shocking that the <u>energy</u> equation should be the same since it is really only the scaling of the volume with time that matters. ~~It does, however~~ The resulting energy equation requires some thought, not merely metaphysical but <u>physical</u>, about what it means to say we have "time" in a universe with no net change in the entropy!

<u>NB</u>: We've left k as a constant, the curvature, but otherwise unspecified. Now using

$$H = \frac{\dot R}{R}$$

as a time dependent Hubble "parameter", we have:

$$H^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho \rightarrow R^2\left(H^2 - \frac{8\pi G}{3}\rho\right) = R_0^2\left(H_0^2 - \frac{8\pi G}{3}\rho_0\right)$$

Then we see that k is related to the density, as you would have expected from the field equations. Again, this is <u>without</u> the Λ term. The critical density is defined by

$$\rho_c = \frac{3H_0^2}{8\pi G} \leftrightarrow k = 0 \quad \text{(Note, if } k=0 \text{ now it is always so!)}$$