Orbit determination with the 2-body integrals

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The discovery of Ceres

Giuseppe Piazzi (1746-1826)

On January 1, 1801 G. Piazzi discovered Ceres, the first asteroid.
He could follow up the asteroid in the sky for about 1 month, collecting 21 observations forming an arc of ~ 3 degrees.

- Problem: find in which part of the sky we have to observe to recover Ceres when it is visible again;
- Orbit determination: given the observations of a Solar system body, compute its Keplerian orbit.

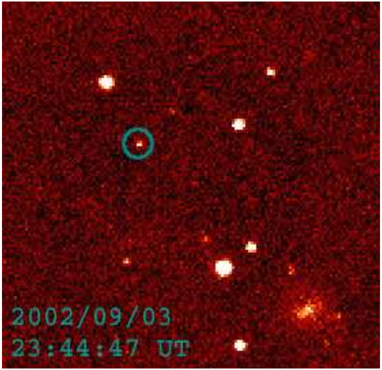
Orbit determination methods

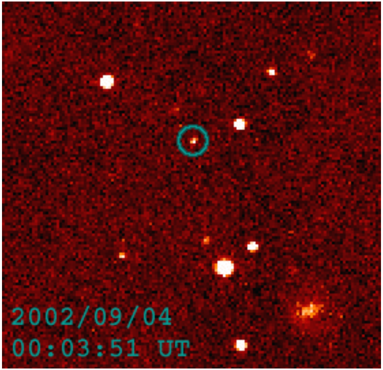
C. F. Gauss proposed a method allowing the recover of Ceres in 1802: he determined an orbit with the observations made by Piazzi. Given at least three observations of a Solar system body, his method consists of two steps:

1. determination of a preliminary orbit;
2. application of the least squares method (also known as differential corrections), using the preliminary orbit as a starting guess.

There are two famous classical methods to define a preliminary orbit from three observations: Laplace's method and Gauss' method. These techniques are still effective today, if the available observations fulfill some requirements.

Modern observations of a Solar system body





Attributables

Given a short arc of observations (*right ascension* and *declination*)

$$(\alpha_i, \delta_i) \in S^1 \times (-\pi/2, \pi/2) \text{ at times } t_i,\ i = 1 \ldots m,$$

of a Solar system body we can compute an attributable

$$\mathcal{A} = (\alpha, \delta, \dot{\alpha}, \dot{\delta}, \bar{t}), \qquad \bar{t} = \frac{1}{m}\sum_i t_i$$

by linear or quadratic interpolation.
The radial distance ρ and the radial velocity $\dot{\rho}$ remains completely undetermined.

Angular Momentum

For a given attributable $\mathcal{A}$ the angular momentum vector can be written as a function of the radial distance and velocity $\rho, \dot{\rho}$:

$$\mathbf{c}(\rho, \dot{\rho}) = \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{D}\dot{\rho} + \mathbf{E}\rho^2 + \mathbf{F}\rho + \mathbf{G}$$

where

$$\begin{aligned} \mathbf{D} &= \mathbf{q} \times \hat{\boldsymbol{\rho}} \\ \mathbf{E} &= \dot{\alpha}\hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\rho}}_{\boldsymbol{\alpha}} + \dot{\delta}\hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\rho}}_{\boldsymbol{\delta}} = \eta\mathbf{n} \\ \mathbf{F} &= \dot{\alpha}\mathbf{q} \times \hat{\boldsymbol{\rho}}_{\boldsymbol{\alpha}} + \dot{\delta}\mathbf{q} \times \hat{\boldsymbol{\rho}}_{\boldsymbol{\delta}} + \hat{\boldsymbol{\rho}} \times \dot{\mathbf{q}} \\ \mathbf{G} &= \mathbf{q} \times \dot{\mathbf{q}} \end{aligned}$$

depend only on the attributable $\mathcal{A}$ and on the motion of the observer $\mathbf{q}, \dot{\mathbf{q}}$ at time $\bar{t}$.

Energy

Given $\mathcal{A}$, we can also write the two-body energy as a function of $\rho, \dot{\rho}$:

$$2\mathcal{E}(\rho,\dot{\rho}) = \dot{\rho}^2 + c_1\dot{\rho} + c_2\rho^2 + c_3\rho + c_4 - \frac{2k^2}{\sqrt{\rho^2 + c_5\rho + c_0}}$$

where

$$\begin{array}{ll} c_0 = |\mathbf{q}|^2 & c_3 = 2(\dot{\alpha} < \dot{\mathbf{q}}, \hat{\boldsymbol{\rho}}_\alpha > + \dot{\delta} < \dot{\mathbf{q}}, \hat{\boldsymbol{\rho}}_\delta >) \\ c_1 = 2 < \dot{\mathbf{q}}, \hat{\boldsymbol{\rho}} > & c_4 = |\dot{\mathbf{q}}|^2 \\ c_2 = \eta^2 & c_5 = 2 < \mathbf{q}, \hat{\boldsymbol{\rho}} > \end{array}$$

depend only on $\mathcal{A}, \mathbf{q}, \dot{\mathbf{q}}$.

Equating the angular momentum

Given two attributables $\mathcal{A}_1, \mathcal{A}_2$ at times t_1, t_2, by equating the angular momentum at the two times we obtain the relation

$$\mathbf{D}_1\dot{\rho}_1 - \mathbf{D}_2\dot{\rho}_2 = \mathbf{J}(\rho_1, \rho_2) \quad (1)$$

where

$$\mathbf{J}(\rho_1, \rho_2) = \mathbf{E}_2\rho_2^2 - \mathbf{E}_1\rho_1^2 + \mathbf{F}_2\rho_2 - \mathbf{F}_1\rho_1 + \mathbf{G}_2 - \mathbf{G}_1 \ .$$

By scalar multiplication of (1) with $\mathbf{D}_1 \times \mathbf{D}_2$ we perform elimination of the variables $\dot{\rho}_1, \dot{\rho}_2$ and obtain the equation

$$q(\rho_1, \rho_2) \stackrel{def}{=} \mathbf{D}_1 \times \mathbf{D}_2 \cdot \mathbf{J}(\rho_1, \rho_2) = 0$$

Equating the energy

Given $\mathcal{A}_1, \mathcal{A}_2$ we can also equate the corresponding two-body energies $\mathcal{E}_1, \mathcal{E}_2$; by vector multiplication of (1) with $\mathbf{D}_1$ and $\mathbf{D}_2$, projecting on $\mathbf{D}_1 \times \mathbf{D}_2$ we obtain

$$\dot{\rho}_1(\rho_1, \rho_2) = \frac{(\mathbf{J} \times \mathbf{D}_2) \cdot (\mathbf{D}_1 \times \mathbf{D}_2)}{|\mathbf{D}_1 \times \mathbf{D}_2|^2} ; \qquad \dot{\rho}_2(\rho_1, \rho_2) = \frac{(\mathbf{J} \times \mathbf{D}_1) \cdot (\mathbf{D}_1 \times \mathbf{D}_2)}{|\mathbf{D}_1 \times \mathbf{D}_2|^2}$$

and, substituting into $\mathcal{E}_1 = \mathcal{E}_2$,

$$\mathcal{F}_1(\rho_1, \rho_2) - \frac{2k^2}{\sqrt{\mathcal{G}_1(\rho_1)}} = \mathcal{F}_2(\rho_1, \rho_2) - \frac{2k^2}{\sqrt{\mathcal{G}_2(\rho_2)}} .$$

By squaring twice we obtain the polynomial equation

$$p(\rho_1, \rho_2) \overset{def}{=} \left[(\mathcal{F}_1 - \mathcal{F}_2)^2 \mathcal{G}_1 \mathcal{G}_2 - 4k^4 (\mathcal{G}_1 + \mathcal{G}_2)\right]^2 - 64k^8 \mathcal{G}_1 \mathcal{G}_2 = 0$$

with total degree 24.

Intersections between the curves

Idea of equating the angular momentum and energy to compute an orbit: *Taff and Hall (1976)*.

We study the problem

$$\begin{cases} p(\rho_1, \rho_2) = 0 \\ q(\rho_1, \rho_2) = 0 \end{cases} , \qquad \rho_1, \rho_2 > 0 \tag{2}$$

with classical Algebraic Geometry methods. We can write

$$p(\rho_1, \rho_2) = \sum_{j=0}^{20} a_j(\rho_2)\ \rho_1^j ,$$
$$q(\rho_1, \rho_2) = b_2\ \rho_1^2 + b_1\ \rho_1 + b_0(\rho_2)$$

for some coefficients a_i, b_j, depending only on ρ_2.

Elimination of ρ_1

We consider the resultant $Res(\rho_2) = \mathtt{res}(p, q, \rho_1)(\rho_2)$ of p, q with respect to ρ_1: it is a 48 degree polynomial defined as the determinant of the Sylvester matrix

$$\mathtt{Sylv}(\rho_2) = \begin{pmatrix} a_{20} & 0 & b_2 & 0 & \dots & \dots & 0 \\ a_{19} & a_{20} & b_1 & b_2 & 0 & \dots & 0 \\ \vdots & \vdots & b_0 & b_1 & b_2 & \dots & \vdots \\ \vdots & \vdots & 0 & b_0 & b_1 & \dots & \vdots \\ a_0 & a_1 & \vdots & \vdots & \vdots & b_0 & b_1 \\ 0 & a_0 & 0 & 0 & 0 & 0 & b_0 \end{pmatrix}.$$

The ρ_2 component of a solution of (2) must be a root of $Res(\rho_2)$.

Computation of the solutions

- Evaluation of $a_i(\rho_2), b_j(\rho_2)$ at the 64-th roots of unit $\omega_k = e^{-2\pi i \frac{k}{64}}$, $k = 0, .., 63$ by a DFT algorithm.
- Computation of the determinant of the 64 Sylvester matrices; by relation

$$\det\left(\mathtt{Sylv}(\rho_2)|_{\rho_2=\omega_k}\right) = \left(\det\mathtt{Sylv}(\rho_2)\right)|_{\rho_2=\omega_k}$$

  we have the values of $Res(\rho_2)$ at the 64-th roots of unit.
- Application of an IDFT algorithm to obtain the coefficients of $Res(\rho_2)$ from its evaluations.

Computation of the solutions

- Computation of the positive roots $\rho_2(k)$ of $Res(\rho_2)$.
- For each k solve $q(\rho_1, \rho_2(k)) = 0$ and compute the two possible values for $\rho_1(k,1), \rho_1(k,2)$.
- Compute $p(\rho_1(k,1), \rho_2(k)), p(\rho_1(k,2), \rho_2(k))$ and select the pair that gives the smaller absolute value.
- Compute the corresponding values of $\dot{\rho}_1, \dot{\rho}_2$.
- Change coordinates to obtain the related Keplerian orbits at times $\tilde{t}_i = \bar{t}_i - \frac{\rho_i}{c}, i = 1, 2$, corrected by aberration.
- Eliminate *spurious solutions* in the previous steps.

Selection of the solutions

The knowledge of the angular momentum vector and of the energy at a given time allows us to compute the elements

$$a, e, I, \Omega \; .$$

The two attributables $\mathcal{A}_1, \mathcal{A}_2$ gives 8 scalar data, thus the problem is overdetermined: from a non-spurious solution ρ_1, ρ_2 of (2) we obtain the same values of a, e, I, Ω at times $\tilde{t}_1, \tilde{t}_2$, but we must check for the *compatibility conditions*

$$\omega_1 = \omega_2 \, , \qquad \ell_1 = \ell_2 + n(\tilde{t}_1 - \tilde{t}_2) \, ,$$

where n is the *mean motion* of the celestial body.

Covariance of the solutions

Given $\mathbf{A} = (\mathcal{A}_1, \mathcal{A}_2)$ with covariance matrices $\Gamma_{\mathcal{A}_1}, \Gamma_{\mathcal{A}_2}$, let

$$\mathbf{R} = \mathbf{R}(\mathbf{A}) = (\mathcal{R}_1(\mathbf{A}), \mathcal{R}_2(\mathbf{A}))\,, \qquad \mathcal{R}_i = (\rho_i, \dot{\rho}_i), \quad i = 1, 2$$

be a solution of

$$\mathbf{\Phi}(\mathbf{R}; \mathbf{A}) = 0\,, \qquad \mathbf{\Phi}(\mathbf{R}; \mathbf{A}) = \left(\begin{array}{c} \mathbf{D}_1 \dot{\rho}_1 - \mathbf{D}_2 \dot{\rho}_2 - \mathbf{J}(\rho_1, \rho_2) \\ \mathcal{E}_1(\rho_1, \dot{\rho}_1) - \mathcal{E}_2(\rho_2, \dot{\rho}_2) \end{array} \right)\,.$$

If both $(\mathcal{A}_1, \mathcal{R}_1(\mathbf{A}))$, $(\mathcal{A}_2, \mathcal{R}_2(\mathbf{A}))$ give bounded orbits at times

$$\tilde{t}_i = \tilde{t}_i(\mathbf{A}) = \bar{t}_i - \frac{\rho_i(\mathbf{A})}{c}\,, \qquad i = 1, 2\,,$$

then we can compute the corresponding Keplerian elements.

Covariance of the solutions

We consider the vector

$$\Delta_{1,2} = (\Delta\omega, \Delta\ell)\,,$$

representing the difference in perihelion argument and mean anomaly of the two orbits, comparing the anomalies at the same time $\tilde{t}_1$. We introduce the map

$$(\mathcal{A}_1, \mathcal{A}_2) = \mathbf{A} \mapsto \boldsymbol{\Psi}(\mathbf{A}) = (\mathcal{A}_1, \mathcal{R}_1, \Delta_{1,2})\,,$$

giving the orbit $(\mathcal{A}_1, \mathcal{R}_1(\mathbf{A}))$ in spherical coordinates at time $\tilde{t}_1$, together with the difference $\Delta_{1,2}(\mathbf{A})$ in the angular elements, which are not constrained by the angular momentum and the energy integrals.

Covariance propagation

By the covariance propagation rule

$$\Gamma_{\mathbf{\Psi}(\mathbf{A})} = \frac{\partial \mathbf{\Psi}}{\partial \mathbf{A}} \, \Gamma_{\mathbf{A}} \left[\frac{\partial \mathbf{\Psi}}{\partial \mathbf{A}}\right]^T ,$$

where

$$\frac{\partial \mathbf{\Psi}}{\partial \mathbf{A}} = \begin{bmatrix} I & 0 \\ \frac{\partial \mathcal{R}_1}{\partial \mathcal{A}_1} & \frac{\partial \mathcal{R}_1}{\partial \mathcal{A}_2} \\ \frac{\partial \Delta_{1,2}}{\partial \mathcal{A}_1} & \frac{\partial \Delta_{1,2}}{\partial \mathcal{A}_2} \end{bmatrix} \quad \text{and} \quad \Gamma_{\mathbf{A}} = \begin{bmatrix} \Gamma_{\mathcal{A}_1} & 0 \\ 0 & \Gamma_{\mathcal{A}_2} \end{bmatrix} .$$

The matrices $\frac{\partial \mathcal{R}_i}{\partial \mathcal{A}_j}, i, j = 1, 2$, can be computed from

$$\frac{\partial \mathbf{R}}{\partial \mathbf{A}}(\mathbf{A}) = -\left[\frac{\partial \mathbf{\Phi}}{\partial \mathbf{R}}(\mathbf{R}(\mathbf{A}), \mathbf{A})\right]^{-1} \frac{\partial \mathbf{\Phi}}{\partial \mathbf{A}}(\mathbf{R}(\mathbf{A}), \mathbf{A}) .$$

Identification of attributables (*linkage*)

Problem: decide if the two sets of observations defining $\mathcal{A}_1$, $\mathcal{A}_2$ can belong to the same Solar system body.
We need to check whether the failure of condition

$$\Delta_{1,2}(\mathbf{A}) = \mathbf{0}$$

is within an acceptable range of values, compatible with the observational errors. Take the marginal covariance matrix:

$$\Gamma_{\Delta_{1,2}} = \frac{\partial \Delta_{1,2}}{\partial \mathbf{A}} \Gamma_{\mathbf{A}} \left[\frac{\partial \Delta_{1,2}}{\partial \mathbf{A}} \right]^T ;$$

the inverse matrix $C^{\Delta_{1,2}} = \Gamma^{-1}_{\Delta_{1,2}}$ defines a norm $\| \cdot \|_\star$ in the $(\Delta\omega, \Delta\ell)$ plane, allowing to test the identification of $\mathcal{A}_1, \mathcal{A}_2$:

$$\|\Delta_{1,2}\|^2_\star = \Delta_{1,2} C^{\Delta_{1,2}} \Delta^T_{1,2} \leq \chi^2_{max} , \qquad \chi_{max} \text{ control parameter.}$$

Uncertainty of the orbits

This method also allows to assign an uncertainty to the preliminary orbits that we compute. The solution $(\mathcal{A}_1, \mathcal{R}_1(\mathbf{A}))$, in spherical coordinates, has the marginal covariance matrix

$$\Gamma_{(\mathcal{A}_1,\mathcal{R}_1(\mathbf{A}))} = \begin{bmatrix} \Gamma_{\mathcal{A}_1} & \Gamma_{\mathcal{A}_1,\mathcal{R}_1} \\ \Gamma_{\mathcal{R}_1,\mathcal{A}_1} & \Gamma_{\mathcal{R}_1} \end{bmatrix},$$

with

$$\Gamma_{\mathcal{A}_1} = \frac{\partial \mathcal{A}_1}{\partial \mathbf{A}} \Gamma_{\mathbf{A}} \left[\frac{\partial \mathcal{A}_1}{\partial \mathbf{A}}\right]^T, \qquad \Gamma_{\mathcal{R}_1} = \frac{\partial \mathcal{R}_1}{\partial \mathbf{A}} \Gamma_{\mathbf{A}} \left[\frac{\partial \mathcal{R}_1}{\partial \mathbf{A}}\right]^T,$$

$$\Gamma_{\mathcal{A}_1,\mathcal{R}_1} = \Gamma_{\mathcal{A}_1} \left[\frac{\partial \mathcal{R}_1}{\partial \mathcal{A}_1}\right]^T, \qquad \Gamma_{\mathcal{R}_1,\mathcal{A}_1} = \Gamma^T_{\mathcal{A}_1,\mathcal{R}_1}.$$

Conclusions

- The recent improvements in the observational techniques are stimulating the research in Celestial Mechanics, in particular about the determination of the orbits: the huge amount of astrometric data next to be produced needs to be processed by efficient methods.
- Classical orbit determination methods are still worth to be investigated: they still hide some unsolved features.
- It is interesting to investigate orbit determination methods different from the classical ones, for applications to modern sets of data.